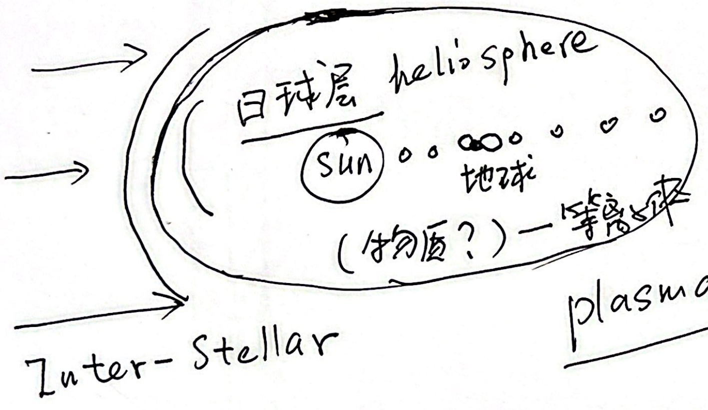
空间等离子体物理学 2022.10.6

1.1 太空？物质？



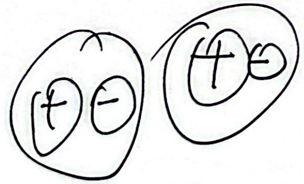
plasma？

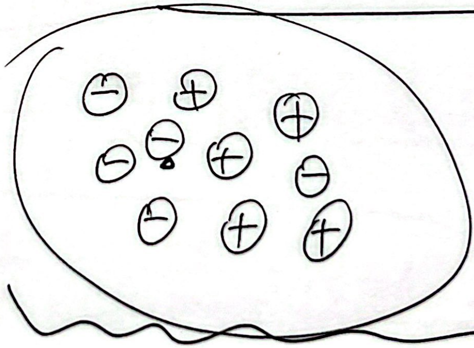
自然界中的 plasma？

三态：固、液、气（分子）—电离→ ？电性（库仑作用）［等离子体］

束缚态

第四态





多体、長程库仑作用

温度：平均运动能量

$\frac{3k_B}{2}T$ （K, eV）

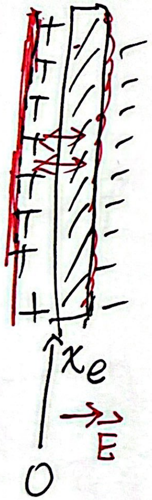
$1\,eV \sim 11600\,K$
$1\,keV \sim 10\,MK$
⇒ 电离（复合）

⇒ 集体行为
振荡现象
屏蔽现象
库仑碰撞

1.2 集体行为的几种表现与等离子体判据

1.2.1 等离子体振荡 (plasma oscillation), Langmuir 波/模 .

$n_p = n_e = n_0$



$$\left.\begin{aligned} m_e\frac{d^2x_e}{dt^2} &= qE = -eE \\ E = \frac{\sigma}{\varepsilon_0} &= \frac{n x_e e}{\varepsilon_0} \end{aligned}\right\} \Rightarrow \frac{d^2x_e}{dt^2} = -\frac{ne^2}{m\varepsilon_0}x_e = -\omega_{pe}^2 x_e$$

$$\boxed{\omega_{pe}^2 = \frac{ne^2}{m_e\varepsilon_0}} \qquad \boxed{\sim \cos(\omega_{pe} x_e)}$$

$$\omega_{pe} \propto \sqrt{n}$$

$q_i = Z_i e$, m_i

$$\Rightarrow \boxed{\omega_{pi}^2 = \frac{n Z_i^2 e^2}{m_i \varepsilon_0}}$$

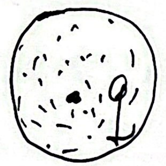
$$\omega_p^2 = \omega_{pe}^2 + \omega_{pi}^2 = \sum_\alpha \omega_{p\alpha}^2 \quad (\text{第四题}).$$

$$\boxed{\omega_{pe}^2 \propto \frac{1}{m_e}, \quad \omega_{pi}^2 \propto \frac{1}{m_i}, \quad \frac{m_p}{m_e} \sim 1836.}$$

$$f_{pe} = \frac{\omega_{pe}}{2\pi} \doteq \cancel{9k}\ \cancel{9000}\ 9\sqrt{n_e(\mathrm{cm}^{-3})}\ \mathrm{kHz}$$

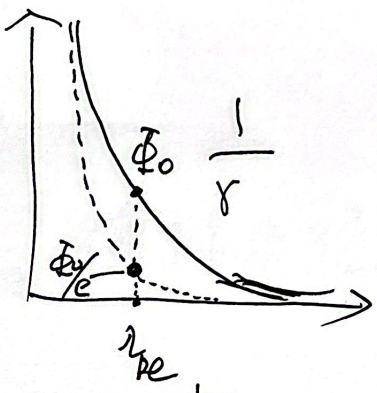
太阳大气 $10^8\ \mathrm{cm}^{-3}$ $\quad f_{pe} \sim 90\ \mathrm{MHz}$

1.2.2 德拜屏蔽 (Debye Shielding)



$$\Phi_0 = \frac{q}{4\pi\varepsilon_0 r} \propto \frac{1}{r}$$

有效屏蔽的范围.



$1/e$

Φ_0 势场中的热平衡分布函数

$$\boxed{n_e = n_{e0} \exp\left(\frac{e\Phi(r)}{k_B T_e}\right)} \sim n_{e0}\left(1 + \frac{e\Phi(r)}{k_B T_e}\right).$$

$\boxed{\text{离子/质子不动}}$

$e\Phi(r) \ll k_B T_e$

势能 粒子运动能量

$\nabla \cdot \vec{E} = \rho_q/\varepsilon_0 = -\frac{e n_e}{\varepsilon_0}$

$\vec{E} = -\nabla\Phi$

$\Rightarrow \nabla^2\Phi = \frac{e\delta n_e}{\varepsilon_0}$

$\rho_q = -en_e + n_p e$

$= \delta n_e \cdot e$

$$\Rightarrow \nabla^2\Phi = \frac{e^2 n_0 \Phi(r)}{k_B T_e \varepsilon_0} = \frac{\Phi(r)}{\lambda_{De}^2}$$

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{d\Phi}{dr}\right) = \frac{\Phi(r)}{\lambda_{De}^2}$$

令 $\Phi(r) = u(r)/r \Rightarrow \frac{d^2 u(r)}{dr^2} = \frac{u(r)}{\lambda_{De}^2}$

$\Rightarrow u(r) = u_0 \exp\left(-\frac{r}{\lambda_{De}}\right) + u_1$

边界条件：$\underline{r \to 0}$，$\Phi = \Phi_0 = \dfrac{q}{4\pi\varepsilon_0 r} \Rightarrow \underset{r\to 0}{u(r)} = \Phi(r)\cdot r = \underline{\dfrac{q}{4\pi\varepsilon_0}}$

$\underline{r \to \infty}$，$\Phi \to 0 \Rightarrow u(r\to\infty) = 0$

代入，得：$\boxed{\Phi(r) = \Phi_0 \exp\left(-\dfrac{r}{\lambda_{De}}\right)}$，$\lambda_{De}$ 德拜半径/长度.

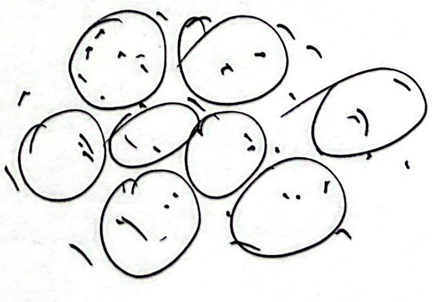
$\Lambda = \frac{4\pi}{3}\lambda_{De}^3 n_0$（等离子体参数），$\underline{\Lambda \gg 1}$

$\underline{\lambda_{De}^2} = \dfrac{k_B T_e \varepsilon_0}{n_0 e^2}$，$\lambda_{Di}^2 = \dfrac{k_B T_i \varepsilon_0}{n_0 z_i^2 e^2}$，$\boxed{\lambda_D^{-2} = \lambda_{De}^{-2} + \lambda_{Di}^{-2}}$

$\boxed{\text{若 } T_e \text{ 越高，则 } \lambda_{De} \text{ 越大？}}$

尺度：准中性成立的尺度（实现电屏蔽的尺度）

$\Lambda \gg 1$



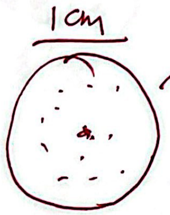
球内：库仑作用

球外：自洽场 → 集体行为

时空匀滑 $E(\vec{r}, t)$

1.2.3 准中性与响应时间

准中性 ： $n_{e0} = \sum_i z_i n_{i0}$.



$$\left.\begin{array}{l} \xrightarrow{} 10^{-4} \text{ 电子} \\ \frac{4}{3}\pi r^3 n / 10^4 \\ n = 10^{14}\ cm^{-3} \end{array}\right\} \Rightarrow E = 6\times10^5\ V/m$$

$4\pi r^2 \cdot E$

响应时间：$\lambda_{De}/v_{the} = \sqrt{\frac{k_B T_e \varepsilon_0}{n_0 e^2}} \Big/ \sqrt{\frac{k_B T_e}{m_e}} = \sqrt{\frac{m_e \varepsilon_0}{n_0 e^2}} = \frac{1}{\omega_{pe}}$.

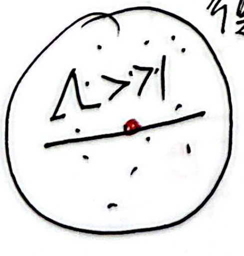
1.2.4 判据

$\Lambda \gg 1$, ($L \gg \lambda_{De}$, $T \gg \tau$), (电性 >> 中性作用)

1.3 库仑碰撞 (经典库仑相互作用).

$\max(\nu_{ee}, \omega_{pe}) \gg \nu_{en}$

德拜球



$\left\{\begin{array}{ll} \text{大角度散射} & \theta \sim \frac{\pi}{2} \\ \text{小角度散射} . & \theta \ll 1 \end{array}\right.$

$q_1 \longrightarrow \quad \cdot\ q_2$

$\tan\left(\frac{\theta_c}{2}\right) = \frac{q_1 q_2}{8\pi\varepsilon_0 E_c \rho} \sim \frac{\theta_c}{2}$

大角度散射截面：$\frac{\pi \rho_L^2}{4}$，$\boxed{\frac{\rho_L}{2} = \frac{q_1 q_2}{8\pi\varepsilon_0 E_c}}$ $\sim \sigma_N$ $\sim \sigma_F$

λ_{De} $\frac{\rho_L}{2}$

$\frac{\rho_L}{2}$，λ_{De}.

$\lambda_{De} >> \rho_L$

(1)

面：$\sim \pi\lambda_{De}^2$

$\overline{\sigma_F} = \sigma_F / N$

$\Delta\theta_{ij} = \frac{q_1 q_2}{8\pi\varepsilon_0 E_c \rho}$

某一次运碰撞(i)

球内 Λ 个粒子.

$\Delta\theta_i = \Delta\theta_{i1} + \Delta\theta_{i2} + \cdots + \Delta\theta_{i\Lambda}$

$j = 1, \Lambda$

$\overline{\theta_c} = \sqrt{\sum \theta_i^2} = \left[\frac{1}{\pi\lambda_D^2}\int_{\frac{\rho_L}{2}}^{\lambda_D} \Delta\theta_{ij}^2 \, 2\pi\rho \, d\rho\right]^{1/2}$

$\Delta\theta_{ij}^2 = \frac{q_1^2 q_2^2}{8^2\pi^2\varepsilon_0^2 E_c^2 \rho^2}$

$= \left[\frac{\frac{q_1^2 q_2^2}{8\pi\varepsilon_0^2 E_c^2}\ln\left(\lambda_D / \frac{\rho_L}{2}\right)}{\pi\lambda_D^2}\right]^{1/2}$

$\sqrt{N}$, $\overline{\theta_c} = \frac{\pi}{2} \Rightarrow N = \left(\frac{\pi}{2}/\overline{\theta_c}\right)^2 = \cancel{\frac{\pi^2}{4}\frac{16\pi^2\varepsilon_0^2 E_c^2 \rho^2}{q_1^2 q_2^2}}$ $\frac{\pi^2}{4}\frac{\pi\lambda_D^2}{\frac{q_1^2 q_2^2}{8\pi\varepsilon_0^2 E_c^2}\ln(\lambda_D / \frac{\rho_L}{2})}$.

$\cancel{\sum_{i=1}^{N}\left(\sum_{j=1}^{\Lambda}\Delta\theta_{ij}\right)^2 = N\,\overline{\theta_i}^2}$

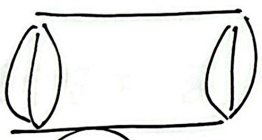
$\boxed{\frac{\lambda_D}{\frac{\rho_L}{2}} = 6\Lambda}$

$\overline{\sigma_F} = \frac{\pi\lambda_D^2}{N} \sim \frac{4\cancel{\lambda_D^2}}{\pi}\frac{q_1^2 q_2^2}{8\pi\varepsilon_0^2 E_c^2}\frac{\ln(6\Lambda)}{\cancel{\lambda_D^2}}$ 与 $\sigma_N \sim \frac{\pi\rho_L^2}{4}$ 比较

$= \frac{32}{\pi^2}\sigma_N \ln 6\Lambda. \Rightarrow \sigma_N << \overline{\sigma_F}$

$\lambda_f \overline{\sigma_F} n = 1$

$\tau = \frac{\lambda_f}{v_{the}}$, $\nu = v_{the}/\lambda_f = v_{the} \cdot n \cdot \overline{\sigma_F}$



$\lambda_f \cdot \overline{\sigma_F} \cdot n = 1$.

$\nu \propto \sqrt{T} \cdot n / T^2 \sim \boxed{n T^{-3/2}}$

对于高温、低密度等离子体，常可忽略库仑碰撞 $\Longrightarrow$ 无碰撞等离子体

Collisionless Plasma.

1.4 等离子体中的辐射.

1.5 描述方法：粒子模拟、单粒子轨道、磁流体力学（MHD）

动力理论

（单流体）、双流体

流体力学近似.

f 速度分布函数.

$n, \bar{v}, T$

相空间.

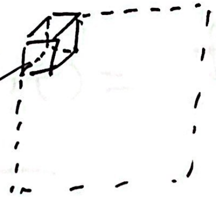
1.5 数学描述方法

多粒子体系（自由电荷）：

$$\begin{cases} \vec{F} = m_i\vec{a} = m_i\dfrac{d\vec{v}}{dt} = q_i(\vec{E}+\vec{v}\times\vec{B}) & , i=1,\cdots,N \\ \nabla\cdot\vec{E} = \rho_q/\varepsilon_0 & \rho_q = \sum_\alpha n_\alpha q_\alpha \\ \nabla\cdot\vec{B} = 0 & \\ \nabla\times\vec{E} = -\partial\vec{B}/\partial t & \\ \nabla\times\vec{B} = \mu_0\vec{j} + \mu_0\varepsilon_0\dfrac{\partial\vec{E}}{\partial t} & \vec{j} = \sum_\alpha n_\alpha q_\alpha \vec{v}_\alpha \end{cases}$$

① 给定 $\vec{E}$, $\vec{B}$，分析~~单~~单粒子轨道

② 粒子模拟（宏粒子，



cell

x_i, y_j, z_k

particle-in-cell

$1000\times1000\times1000$

10^{10}

计算资源 ⟶ 研究小尺度的物理问题

? 电子，质子　$m_p/m_e \sim 1845$

（粒子动力学）

kinetic scale

动理学尺度 v.s. 流体力学尺度

小/微观　大/宏观

第三章 ③ 流体力学近似

近似条件：碰撞足够频繁，成团

n, $\vec{v}$, T, $p \sim nkT\ (\rho RT)$.

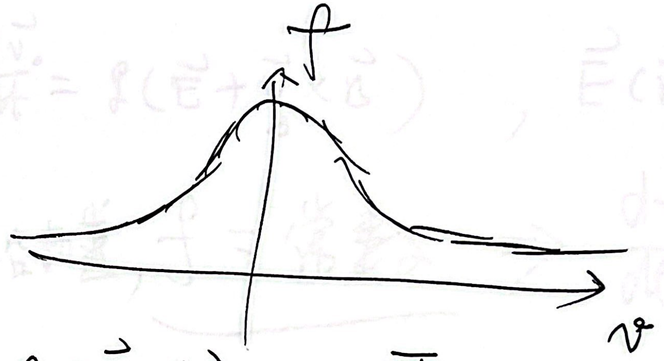
磁流体力学.

$$\begin{cases} \dfrac{\partial n}{\partial t} + \nabla\cdot(n\vec{v}) = 0 & \text{质量守恒} \\ \rho\dfrac{d\vec{v}}{dt} = -\nabla p + \vec{j}\times\vec{B} + \rho\vec{g} \end{cases}$$

④ 动理论 (Kinetic Theory)

$f(\vec{r}, \vec{v}, t)$

$d\vec{r}$



$$\int d^3\vec{v}\, f(\vec{r}, \vec{v}, t) = n(\vec{r}, t).$$ 零阶矩

归一化 $\hat{f}(\vec{r}, \vec{v}, t) = f(\vec{r}, \vec{v}, t) / n(\vec{r}, t).$

$\vec{v} = \vec{u} + \vec{w}$

$$\int \vec{v} f(\vec{r}, \vec{v}, t)\, d^3\vec{v} = n\vec{u}(\vec{r}, t)$$ 一阶矩.

$$\int \vec{w} f\, d^3\vec{v} = 0$$

二阶矩 $$\int m\vec{v}\vec{v} f(\vec{r}, \vec{v}, t)\, d^3\vec{v} = \int m(\vec{u}+\vec{w})(\vec{u}+\vec{w}) f\, d^3\vec{v} = \int m\vec{u}\vec{u} f\, d^3\vec{v} + \int m\vec{w}\vec{w} f\, d^3\vec{v}$$

$$\overleftrightarrow{P} = \int m\vec{w}\vec{w} f\, d^3\vec{v}$$

$$= mn\vec{u}\vec{u} + \overleftrightarrow{P}$$

$\vec{r}, \vec{v}$, $d\vec{r}\,d\vec{v}$ 六维相空间

$f(\vec{r}, \vec{v}, t)$ phase space

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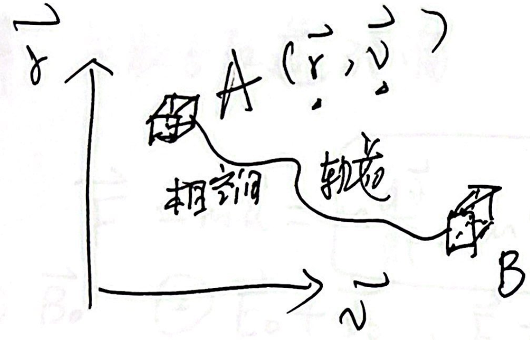


$$\vec{F} = m\frac{d\vec{v}}{dt} = q(\vec{E}+\vec{v}\times\vec{B}), \quad \vec{E}(\vec{r},t), \vec{B}(\vec{r},t)$$

$B(\vec{r},\vec{v}')$. 沿轨道, f = 常数 $\Rightarrow \frac{df}{dt}=0$

$$\frac{d f(\vec{r},\vec{v},t)}{dt} = 0 = \frac{d}{dt} f(\vec{r}(t), \vec{v}(t), t)$$

$$\boxed{\frac{d}{dt} = \frac{\partial}{\partial t} + \vec{v}\cdot\nabla}$$

$$\Rightarrow \frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{d\vec{r}(t)}{dt}\cdot\frac{\partial f}{\partial \vec{r}} + \frac{d\vec{v}(t)}{dt}\cdot\frac{\partial f}{\partial \vec{v}} .$$

$$\boxed{\frac{d\vec{r}(t)}{dt} = \vec{v}, \quad \frac{d\vec{v}}{dt} = \vec{a}} .$$

$\frac{\partial}{\partial \vec{r}} \sim \nabla_{\vec{r}}, \frac{\partial}{\partial \vec{v}} \sim \nabla_{\vec{v}}$

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \vec{v}\cdot\nabla_{\vec{r}} f + \vec{a}\cdot\nabla_{\vec{v}} f = 0$$

$$\boxed{\frac{\partial f_\alpha}{\partial t} + \vec{v}\cdot\nabla_{\vec{r}} f_\alpha + \frac{q_\alpha}{m_\alpha}(\vec{E}+\vec{v}\times\vec{B})\cdot\nabla_{\vec{v}} f_\alpha = 0}$$

(Vlasov 方程)

无碰撞 Boltzmann 方程

$$\left(\frac{\partial f}{\partial t}\right)_c \propto (\nu_i - \nu_j)\nu$$

第二章 单粒子轨道理论

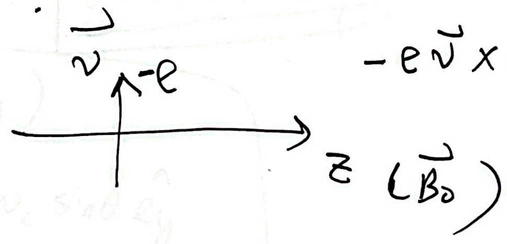
$$\vec{F} = m\vec{a} = \boxed{m\frac{d\vec{v}}{dt} = m\frac{d^2\vec{r}}{dt^2} = q(\vec{E}+\vec{v}\times\vec{B})} = q(\vec{E}(\vec{r},t) + \vec{v}\times\vec{B}(\vec{r},t))$$

① $\vec{B}_0$ ② $\vec{E}_0+\vec{B}_0$, $\vec{F}_0+\vec{B}_0$ ③ $\vec{B} = \vec{B}_0 + \vec{B}_1 + \vec{B}_2$ ④ $\vec{E} = \vec{E}_0 + \vec{E}_1$, $\vec{E}(t)$

2.1 均恒外场中的单粒子运动．

$-e\vec{v}\times\vec{B}$

2.1.1. …磁… ($\vec{B}_0 = B_0\hat{e}_z$)



$$m\frac{d\vec{v}}{dt} = q\vec{v}\times\vec{B}_0$$

$$\Rightarrow \begin{cases} m\dfrac{dv_x}{dt} = q\dfrac{B_0}{m}v_y \\ \dfrac{dv_y}{dt} = -q\dfrac{B_0}{m}v_x \end{cases} \Rightarrow \frac{d^2v_x}{dt^2} = -\left(\frac{qB_0}{m}\right)^2 v_x, \quad \frac{q^2B_0^2}{m^2} = w_c^2, \quad w_c = \frac{qB_0}{m}.\ \Omega_c = \frac{|q|B_0}{m}$$

$$\frac{d^2v_y}{dt^2} = -\left(\frac{qB_0}{m}\right)^2 v_y.$$

$$\left.\begin{aligned} v_x &= v_{\perp 0}\cos(w_ct+\alpha) \\ v_y &= -v_{\perp 0}\sin(w_ct+\alpha). \end{aligned}\right\} \Rightarrow \left\{\begin{aligned} x &= x_0 + \frac{v_{\perp 0}}{w_c}\sin(w_ct+\alpha) \\ y &= y_0 + \frac{v_{\perp 0}}{w_c}\cos(w_ct+\alpha) \end{aligned}\right\} \Rightarrow (x-x_0)^2+(y-y_0)^2 = \left(\frac{v_{\perp 0}}{w_c}\right)^2 = r_c^2$$

$$r_c = \left|\frac{v_{\perp 0}}{w_c}\right| = \frac{v_{\perp 0}}{\Omega_c}$$

对于电子，右旋；质子．左旋

$$\Omega_{ce} = \frac{eB}{m_e} \gg \Omega_{ep} = \frac{eB}{m_p}.$$

$$r_{ce} \sim \frac{v_{\perp 0}}{\Omega_c} \sim \frac{\sqrt{kT/m_e}}{eB/m_e} \propto \sqrt{m_e}$$

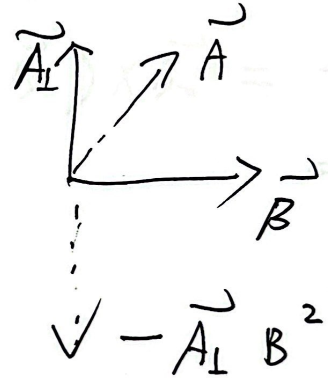
$B = 10G$, $f_{ce} = \Omega_{ce}/2\pi = ?$ $2.8\,MHz\,B(G) = 28\,MHz$.

$$m\frac{d\vec{v}_c}{dt} = q\vec{v}_c \times \vec{B}_0 \Rightarrow \cancel{m}\frac{d\vec{v}_c}{dt} = \frac{q}{m}\left(\frac{d(\vec{r}_c\times\vec{B}_0)}{dt}\right) \Rightarrow \boxed{\vec{v}_c = \frac{q}{m}\,\vec{r}_c\times\vec{B}_0}$$

(12)

$$\vec{v}_c \times \vec{B}_0 = \frac{q}{m}(\vec{r}_c\times\vec{B}_0)\times\vec{B}_0 = -\frac{q}{m}\vec{r}_c B_0^2$$

$$(\vec{A}_\perp\times\vec{B})\times\vec{B} = -\vec{A}_\perp B^2$$

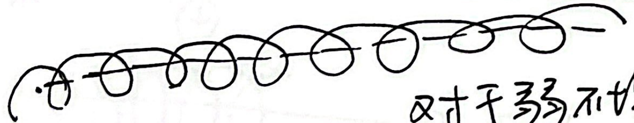


$$\Rightarrow \boxed{\vec{r}_c = -\frac{m}{qB_0^2}\vec{v}_c\times\vec{B}_0}. \quad \boxed{\theta = \omega_c t + \theta_0}$$

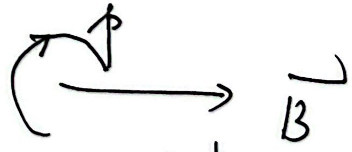
$$\boxed{\begin{aligned}\vec{r}_c &= r_c(\sin\theta\,\hat{e}_x + \cos\theta\,\hat{e}_y)\\ \vec{v}_c &= r_c\omega_c\cos\theta\,\hat{e}_x - r_c\omega_c\sin\theta\,\hat{e}_y\end{aligned}}$$

① Guiding Center:(引导中心);把粒子运动分解成 GC 运动 + 回旋运动.

(1)慢 (2)快



对于弱不均匀场,经常可以



② $\mu = IS = \pi r_c^2 \cdot \frac{q}{T} = \pi r_c^2 \cdot \frac{q\Omega_c}{2\pi} = \frac{1}{2}\frac{v_\perp^2}{\Omega_c^2}\cdot q = \frac{1}{2}\frac{v_\perp^2}{qB}\cancel{q}\, m = \frac{m v_\perp^2}{2B} = W_\perp/B$

$\vec{\mu} = -\frac{W_\perp}{B}\frac{\vec{B}}{B}$ (抗磁介质)

③ 磁场作用:约束,等效磁镜

2.1.2 电场漂移 $\vec{E}_0 + \vec{B}_0$ (drift)

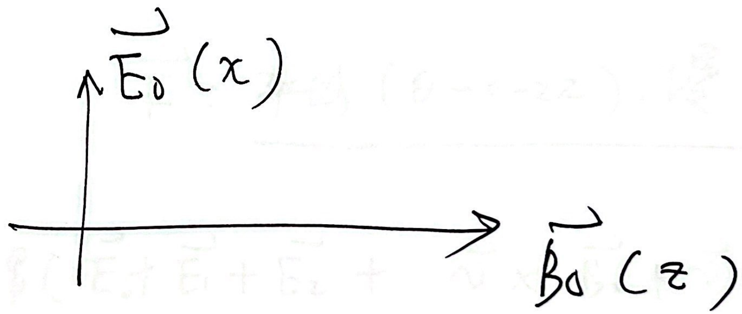
$$m\frac{d\vec{v}}{dt} = q(\vec{E}_0 + \vec{v}\times\vec{B}_0)$$

$$\vec{E}_0 + \vec{v}_D\times\vec{B}_0 = 0 \Rightarrow \vec{v}_D\times\vec{B}_0 = -\vec{E}_0 \Rightarrow (\vec{v}_D\times\vec{B}_0)\times\vec{B}_0 = -\vec{v}_D B_0^2 = -\vec{E}_0\times\vec{B}_0$$

$$\Rightarrow \boxed{\vec{v}_D = \vec{E}_0\times\vec{B}_0 / B_0^2}$$

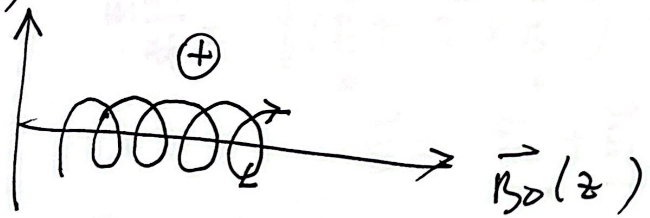
$$\vec{v} = \vec{v}_D + \vec{v}'$$

$$\boxed{m\frac{d\vec{v}'}{dt} = q\vec{v}'\times\vec{B}_0}$$



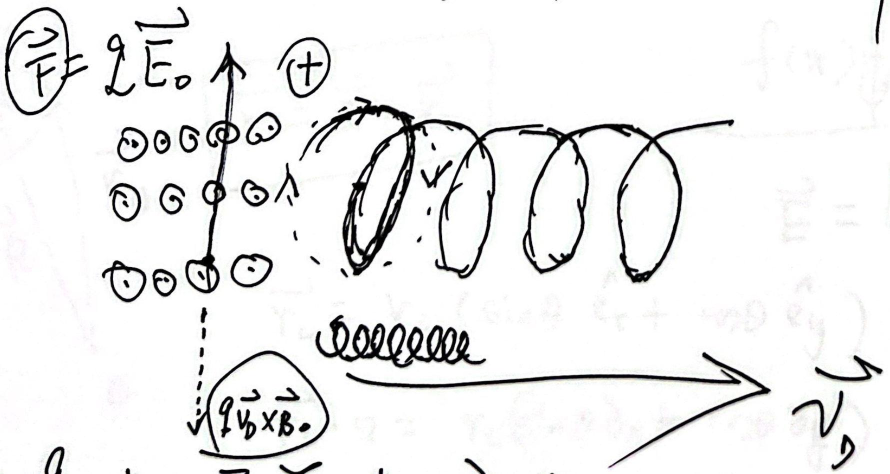
$$\vec{v}_{DF} = \frac{\vec{F}\times\vec{B}_0}{qB_0^2}$$

$\vec{F} = q\vec{E}_0(x)$



$$r_c = \frac{v_\perp}{|\omega_c|}$$

$\vec{F} = q\vec{E}_0$

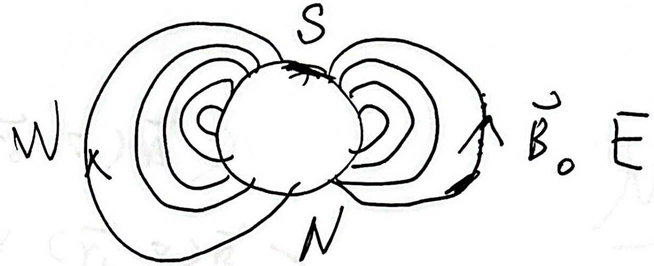


回旋轨迹不闭合 ⇒ 引导中心的（横向）漂移.

漂移 $\vec{v}_{DE}$ 与 q, m 无关！⇒ 整体漂移.

2.1.3 重力场与任意外加力场中的导心漂移

$$\vec{v}_{DG} = \boxed{\frac{m\vec{g}\times\vec{B}_0}{qB_0^2}}$$



$$\vec{v}_{DF} = \frac{\vec{F}\times\vec{B}_0}{qB_0^2}.$$

$\vec{F}$ 定常场与

$$\Uparrow$$

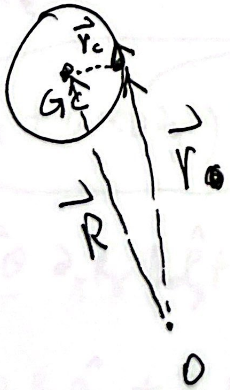
$$\boxed{m\frac{d\vec{v}}{dt} = \vec{F} + q\vec{v}\times\vec{B}_0}$$

$\vec{F}$：平均 $(\theta \sim 0\text{-}2\pi)$. 慢变化 $(\tau \gg \frac{1}{\Omega_c})$

2.2 弱变化磁场中粒子的运动

$$m\frac{d\vec{v}}{dt} = q(\vec{E}_0 + \vec{v}\times\vec{B}_0) = q(\vec{E}_0 + \underbrace{\vec{E}_1 + \vec{E}_2} + \vec{v}\times\vec{B}_0 + \underbrace{\vec{v}\times\vec{B}_1 + \vec{v}\times\vec{B}_2}).$$

$$\vec{E} = \vec{E}_0 + \vec{E}_1 + \vec{E}_2,\quad \vec{B} = \vec{B}_0 + \vec{B}_1 + \vec{B}_2.$$



$$\boxed{\vec{r} = \vec{R} + \vec{r}_c}$$

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2}(x-x_0)^2$$

$$\vec{E} = \vec{E}_0(R) + \vec{r}_c\cdot\nabla\vec{E}\Big|_R + \frac{1}{2}\vec{r}_c(\vec{r}_c\cdot\nabla)\cdot\nabla\vec{E}\Big|_R$$

（自行验证）

$$\vec{r}_c = r_c(\sin\theta\,\hat{e}_x + \cos\theta\,\hat{e}_y)$$

$$\vec{r}_c\cdot\nabla = r_c(\sin\theta\,\partial_x + \cos\theta\,\partial_y)$$

$\langle\vec{F}\rangle$ 表示在回旋周期 $(0\le\theta\le 2\pi)$ 上求平均.

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$$m\frac{d\vec{v}}{dt} = q\vec{v}\times\vec{B} = q\vec{v}\times(\vec{B_0} + \vec{r_c}\cdot\nabla\vec{B}) = \vec{F} + q\vec{v}\times\vec{B_0}$$

$$\mu = \frac{W_\perp}{B}$$

求 $\langle\vec{F}\rangle = \cancel{\langle q\vec{v}\times\vec{B_0}\rangle}\ \langle q\vec{v}\times(\vec{r_c}\cdot\nabla)\vec{B}\rangle$

$\vec{v} = \vec{v}_{DF} + \vec{v}_c$

$$= \langle q(\vec{v}_{DF} + \vec{v}_c)\times(\vec{r_c}\cdot\nabla)\vec{B}\rangle$$

$$= \langle q\,\vec{v}_c\times(\vec{r_c}\cdot\nabla)\vec{B}\rangle = -\mu\nabla B$$

$$\vec{v}_c = \frac{q}{m}\vec{r_c}\times\vec{B_0}$$

$$\langle\vec{F}\rangle = \frac{q^2}{m}\langle(\vec{r_c}\cdot\nabla)\vec{B}\times(\vec{r_c}\times\vec{B_0})\rangle$$

$$\vec{A}\times(\vec{B}\times\vec{C}) = \vec{A}\cdot\vec{C}\,\vec{B} - \vec{A}\cdot\vec{B}\,\vec{C}$$

$$\langle\vec{F}\rangle = -\frac{q^2}{m}\langle(\vec{r_c}\cdot\nabla)\vec{B}\cdot\vec{B_0}\,\vec{r_c} - (\vec{r_c}\cdot\nabla)\vec{B}\cdot\vec{r_c}\,\vec{B_0}\rangle$$

$$= \langle\vec{F_\perp}\rangle + \langle\vec{F_\parallel}\rangle$$

同样，可得：$\langle\vec{F_\parallel}\rangle \sim -\mu\nabla B$

$$\vec{r_c}\cdot\nabla = r_c\sin\theta\,\partial_x + r_c\cos\theta\,\partial_y$$

$$\vec{B} = B_x\hat{e}_x + B_y\hat{e}_y + B_z\hat{e}_z$$

$$\vec{B_0} = B_0\hat{e}_z$$

$$\langle(r_c\sin\theta\,\partial_x + r_c\cos\theta\,\partial_y)B_z B_0(r_c\sin\theta\,\hat{e}_x + r_c\cos\theta\,\hat{e}_y)\rangle$$

$$= \langle r_c^2\sin^2\theta\,\partial_x B_z B_0\hat{e}_x + r_c^2\cos^2\theta\,\partial_y B_z B_0\hat{e}_y\rangle$$

$$= \frac{B_0}{2}r_c^2(\partial_x B_z\hat{e}_x + \partial_y B_z\hat{e}_y) \quad (\langle\sin^2\theta\rangle = 1/2)$$

$$= \frac{B_0}{2}r_c^2\,\nabla_\perp B_z \qquad \nabla_\perp = \partial_x\hat{e}_x + \partial_y\hat{e}_y \quad \nabla_\parallel = \partial_z\hat{e}_z$$

$$\Rightarrow \langle\vec{F_\perp}\rangle = -\frac{q^2}{m}\frac{B_0 r_c^2}{2}\nabla_\perp B_z$$

$$= -\frac{\cancel{q^2}\cancel{B_0}}{\cancel{m}\,2}\frac{v_\perp^2 m^{\cancel{2}}}{\cancel{q^2}B_0^{\cancel{2}}}\nabla_\perp B_z$$

$$= -\frac{mv_\perp^2}{2B_0}\nabla_\perp B_z = -\mu\nabla_\perp B_z \sim -\mu\nabla B$$

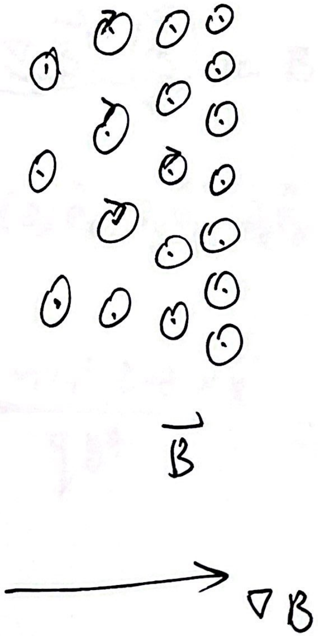
立刻可得 $\vec{v}_{D\nabla} = \frac{-\mu\nabla B \times \vec{B}}{qB^2}$

梯度漂移 — Gradient Drift

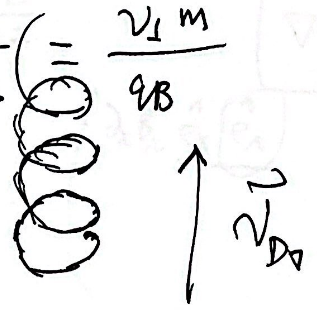
地磁场中
- 离子：自东向西
- 电子：自西向东

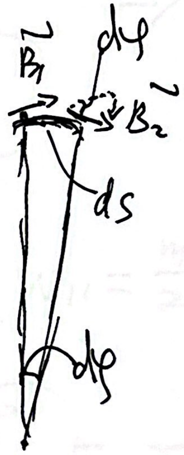
环电流主要成因.

梯度漂移的图像：



$r_c = \frac{v_\perp}{\omega_c} = \frac{v_\perp m}{qB}$





$d\vec{B} = \vec{B}_2 - \vec{B}_1$

$d\varphi = \frac{ds}{R_c} = \frac{dB}{B}$

$\Rightarrow R_c = \frac{ds}{dB} B$

$\Rightarrow \frac{1}{R_c} = \frac{dB}{ds}\frac{1}{B}$

$\frac{dB}{ds} = \frac{\vec{B}}{B} \cdot \nabla B$

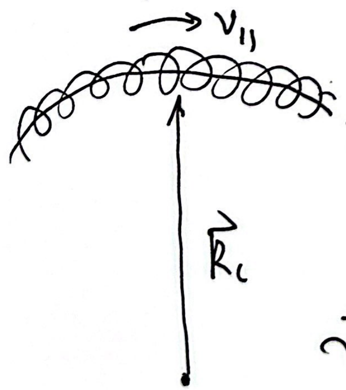
$\Rightarrow \frac{1}{R_c} = \frac{\vec{B}}{B^2} \cdot \nabla B$

$\vec{F}_c = m v_\parallel^2 \frac{\vec{B}}{B^2} \cdot \nabla B$ $\boxed{\frac{\vec{R}_c}{R_c} \sim -\frac{d\vec{B}}{dB}}$

$\frac{\vec{R}_c}{R_c^2} = \frac{1}{R_c}\frac{\vec{R}_c}{R_c} = \frac{\cancel{dB}}{ds}\frac{1}{B} \cdot \left(-\frac{d\vec{B}}{\cancel{dB}}\right)$

$\Leftarrow$ $\vec{F}_c = -\frac{m v_\parallel^2 \vec{B} \cdot \nabla \vec{B}}{B^2}$

2.2.2. 曲率漂移 (Curvature Drift)



$\vec{F}_c = \frac{m v_\parallel^2}{R_c} \frac{\vec{R}_c}{R_c}$

$\vec{v}_{Dc} = \frac{m v_\parallel^2}{R_c^2} \vec{R}_c \times \frac{\vec{B}}{qB^2}$

$$\boxed{\vec{v}_{Dc} = -m v_{\parallel}^2 \frac{\vec{B}\cdot\nabla\vec{B}}{B^2} \times \vec{B} / qB^2}$$

化简 $\vec{B}\cdot\nabla\vec{B}$，条件：$(\nabla\times\vec{B})\times\vec{B} = 0$

$\nabla\times\vec{B} = \mu_0 \vec{j}$

$\Rightarrow \vec{j}\times\vec{B} = 0$ (Lorentz力)

$$(\nabla\times\vec{B})\times\vec{B} = (\vec{B}\cdot\nabla)\vec{B} - \nabla\vec{B}\cdot\vec{B} = 0$$

$$(\nabla\vec{B})\cdot\vec{B} = B\nabla B = \nabla\frac{B^2}{2}$$

$$\boxed{\nabla B^2 = 2\vec{B}\cdot\nabla\vec{B}}$$

$$\nabla\vec{B} = (\partial_x\hat{e}_x, \partial_y\hat{e}_y, \partial_z\hat{e}_z)\begin{pmatrix} B_x\hat{e}_x \\ B_y\hat{e}_y \\ B_z\hat{e}_z \end{pmatrix} = \left(\partial_x B_x \hat{e}_x \hat{e}_x \right.$$

$$\left(W_{\parallel} = \frac{m v_{\parallel}^2}{2} \sim \frac{k_B T}{2},\quad W_{\perp} = \frac{m v_{\perp}^2}{2} \sim k_B T \right)$$

$$\vec{v}_{Dc} = \frac{m v_{\parallel}^2 B\nabla B \times \vec{B}}{qB^4} = -\frac{m v_{\parallel}^2}{qB^3}\nabla B\times\vec{B} \Rightarrow \vec{v}_{D\nabla} = \vec{v}_{DG} + \vec{v}_{Dc} = -\frac{(2W_{\parallel}+W_{\perp})}{qB^3}\nabla B\times\vec{B}$$

$$\frac{dW_\parallel}{dt} = -\frac{d(\mu B)}{dt} = -B\frac{d\mu}{dt} - \mu\frac{dB}{dt}$$

$$m\frac{dv_\parallel}{dt} = -\mu\nabla_\parallel B = -\mu\frac{dB}{dz}$$

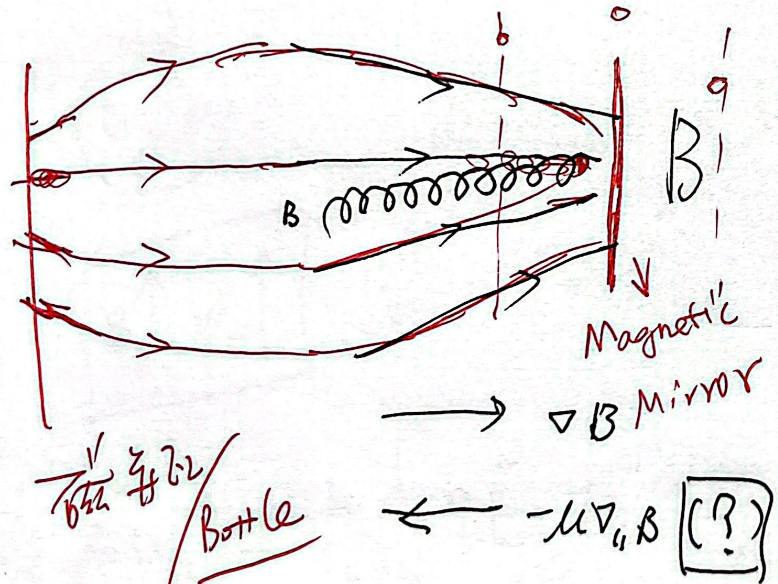
$$\frac{d}{dt}(W_\parallel) = -\mu v_\parallel\frac{\partial B}{\partial z}.$$

$$\Rightarrow -\mu v_\parallel\frac{\partial B}{\partial z} = -B\frac{d\mu}{dt} - \mu\,\underbrace{\vec{v}_\parallel\cdot\nabla B}_{\sim v_\parallel\frac{\partial B}{\partial z}} \Rightarrow \frac{d\mu}{dt} = 0$$

$$\boxed{\mu = \frac{W_\perp\uparrow}{B\uparrow}}$$

Q：有限度

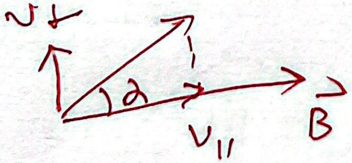
$W_{\perp max} \sim W_0$



反弹点：磁镜点

投掷角 pitch angle (α)

$\alpha >> \alpha_{min} = ?$



逃逸/约束条件：

$$\left.\begin{matrix} B_0 & B_{max} \\ W_{\parallel 0} & \\ W_{\perp 0} & \end{matrix}\right\} \Rightarrow \begin{cases} W_{\parallel 0} + W_{\perp 0} = W_0 \\ \mu = \dfrac{W_\perp}{B} = \text{常量} \end{cases}$$

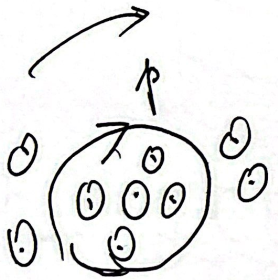
$$\frac{W_0}{B_m} = \frac{W_{\perp 0}}{B_0} \Rightarrow \frac{W_{\perp 0}}{W_0} = \frac{B_0}{B_m}$$

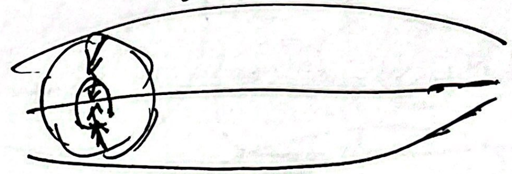
$$\Rightarrow B_m = \left(\frac{W_{\perp 0}}{W_0}\right)^{-1} B_0 = B_0/\sin^2\alpha_0$$

$\sin^2\alpha_0 \geq \frac{B_0}{B_{max}}$. $\boxed{\alpha_0 \geq \arcsin\sqrt{\frac{B_0}{B_{max}}}}$.

② 慢变化 $\tau \gg T_c$, $\left|T_c \frac{\partial B}{\partial t}\right| \ll B$

$\boxed{\text{平行}\delta\,?}$.





$$\frac{dB}{dt} = \frac{\partial B}{\partial t} + \vec{v}\cdot\nabla B$$

$\boxed{B\uparrow,\ W_\perp\uparrow}$. $\frac{\partial \vec{B}}{\partial t} = -\nabla\times\vec{E}$

$$\frac{\Delta W_\perp}{\Delta t} = \frac{1}{T_c}\oint q\vec{E}\cdot d\vec{l} = \frac{q}{T_c}\iint \nabla\times\vec{E}\cdot d\vec{s} = -\frac{q}{T_c}\iint \frac{\partial B}{\partial t}\cdot d\vec{s} = +\frac{q}{T_c} S \frac{\partial B}{\partial t}$$

$$= \mu\frac{\partial B}{\partial t}$$

$W_\perp = \mu B$, $\frac{dW_\perp}{dt} = \mu\frac{dB}{dt} + B\frac{d\mu}{dt} = \mu\frac{\partial B}{\partial t} \Rightarrow \frac{d\mu}{dt} = 0$.

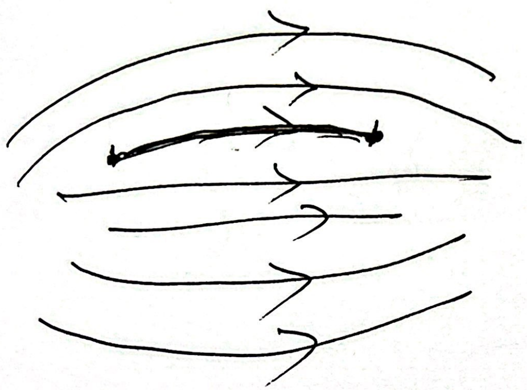
$\mu = W_\perp/B$. 横向($W_\perp$)不变量 . 第一绝热不变量 .

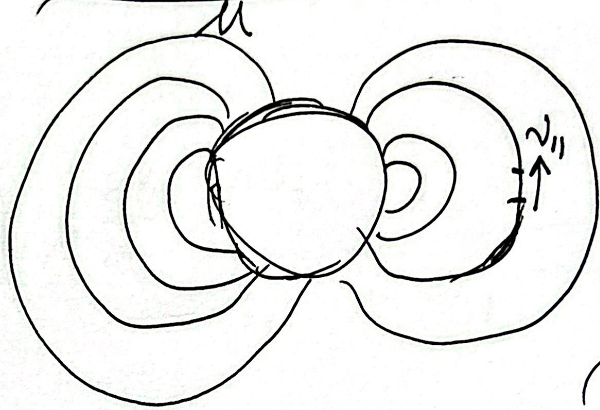
Transverse Invariant   First Adiabatic Invariant

2.2.4 纵向(第二绝热)不变量.

① 回旋、② 反弹、③ 漂移

$\boxed{\text{弱.慢}}$





$$J = \oint v_{\parallel}\, ds$$

$$\frac{dJ}{dt} \approx 0$$

证: $J(\varepsilon, s, t)$ (μ 守恒)

$$\boxed{\varepsilon = \mu B + \frac{m v_{\parallel}^2}{2}} \Rightarrow v_{\parallel} = \pm\sqrt{\frac{2}{m}(\varepsilon - \mu B)}$$

$$J(\varepsilon, s, t) = \int_{s_1}^{s} \sqrt{\frac{2}{m}(\varepsilon - \mu B)}\, ds, \quad \frac{dJ}{dt} = \left.\frac{\partial J}{\partial t}\right|_{\varepsilon, s} + \left(\frac{\partial J}{\partial \varepsilon}\right)_{s,t} \frac{d\varepsilon}{dt} + \cancel{\left(\frac{\partial J}{\partial s}\right)_{\varepsilon,t} \frac{ds}{dt}}$$

$v_{\parallel}$

$$\left.\frac{\partial J}{\partial t}\right|_{\varepsilon, s} = \int_{s_1}^{s} \frac{-\frac{\mu}{m}\frac{\partial B}{\partial t}}{\sqrt{\frac{2}{m}(\varepsilon - \mu B)}}\, ds.$$

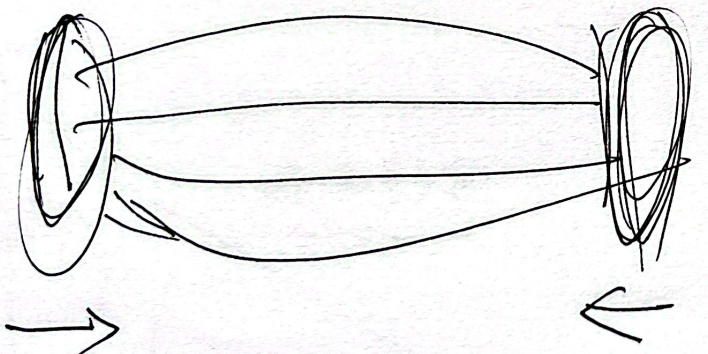
$$\left(\frac{\partial J}{\partial \varepsilon}\right)_{s,t} = \int_{s_1}^{s} \frac{1}{m\sqrt{\frac{2}{m}(\varepsilon - \mu B)}}\, ds$$

$$\frac{d\varepsilon}{dt} = \frac{d(\varepsilon_1 + \varepsilon_2)}{dt} = \cancel{m v_{\parallel} \frac{d v_{\parallel}}{dt}} + \mu \frac{dB}{dt}; \quad \frac{dB}{dt} = \frac{\partial B}{\partial t} + \cancel{(\vec{v}_{\parallel} + \vec{v}_D)} \cdot \nabla B = \frac{\partial B}{\partial t} + \cancel{v_{\parallel}\frac{\partial B}{\partial s}}$$

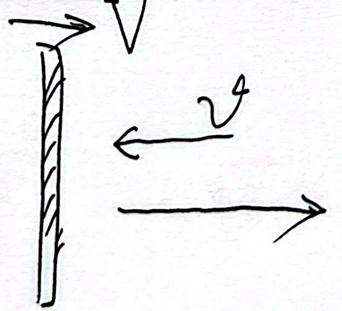
$$\int_A^B J\,ds = J(B) - J(A)$$

$$\Rightarrow \frac{dJ}{dt} = \int_{s_1}^{s_2} \frac{-\frac{\mu}{m}\frac{\partial B}{\partial t}}{v_{\parallel}}\,ds + \mu\frac{\partial B}{\partial t}\int_{s_1}^{s_2}\frac{ds}{m v_{\parallel}} \sim 0 \Rightarrow J是不变量$$

$$J = \oint v_{\parallel}\,ds \sim \overline{v_{\parallel}} \cdot L_B$$



Fermi 加速



在镜子参考系中, $v' = v + V$, $-v' = -v - V$

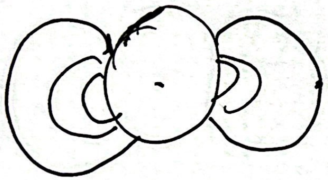
在实验室中, v, $-v-2V$.

$$\Delta E = \frac{m}{2}(v+2V)^2 - \frac{mv^2}{2} = 2m(vV+v^2).$$

3) 第三绝热不变量：Φ 不变量

$\vec{v}_D \sim \vec{B}\times\nabla B$

$\tau \sim \frac{2\pi r}{v_D}$



① μ 不变量

② J 不变量

太阳风存在近似伸加热现象

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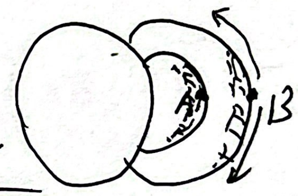
(Fermi 加速)

辐射带

长期存在



$$\vec{F} = q\langle \vec{v}\times\vec{B}_1\rangle = -\mu\nabla B \begin{cases} -\mu\nabla_{\parallel} B \\ -\mu\nabla_{\perp} B \end{cases}$$

$B_A > B_B \Rightarrow L_A < L_B$

$W_{\perp A} > W_{\perp B} \Rightarrow W_{\parallel A} < W_{\parallel B}$

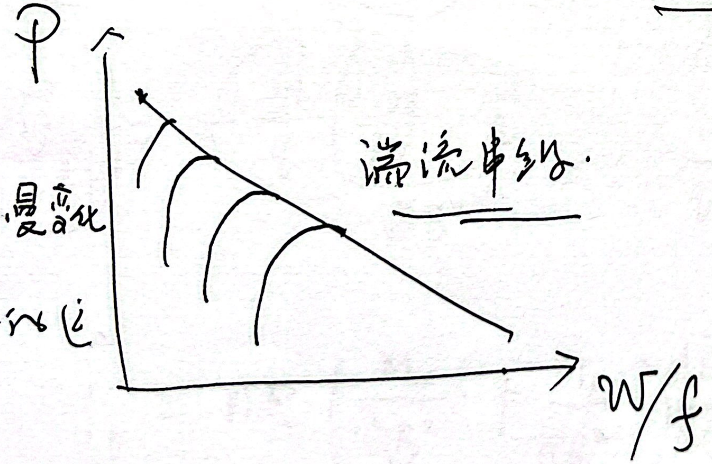
$$\boxed{v_{\parallel A} L_A < v_{\parallel B} L_B}$$

$\cancel{J_A \quad J_B}$

2.3

弱不均匀、慢变化

电场中带电粒子运动



$\langle \sin^2\theta\rangle = \langle\cos^2\theta\rangle = 1/2$

$$\vec{v}_{DF} = \frac{\vec{F}\times\vec{B}}{qB^2}$$

$$\langle q\vec{E}_2\rangle = \langle q\frac{\vec{r}_c}{2}\cdot(\vec{r}_c\cdot\nabla)\cdot\nabla\vec{E}\rangle \sim \frac{q}{4}r_c^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\vec{E}$$

$$\sim \frac{q}{4}r_c^2\nabla_{\perp}^2\vec{E}$$

$\vec{r}_c = r_c(\sin\theta\,\hat{e}_x + \cos\theta\,\hat{e}_y)$

$$\vec{v}_{DE} = \cancel{\frac{\vec{E}\times}{B^2}}\Big(1+\underbrace{\frac{r_c^2}{4}\nabla_\perp^2}_{\text{修正项}}\Big)\frac{\vec{E}\times\vec{B}}{B^2}.$$

慢变化情况：

$$\cancel{\frac{d\vec{v}}{dt}=\frac{q}{m}}$$

$$\frac{d\vec{v}}{dt} = \frac{q}{m}\left[\vec{E}(t)+\vec{v}\times\vec{B}\right] \qquad (\vec{A}\times\vec{B})\times\vec{B} = -\vec{A}_\perp B^2$$

$$\vec{v}_{DE} = \frac{\vec{E}(t)\times\vec{B}}{B^2} \qquad \vec{v} = \vec{v}_c + \vec{v}_{DE}$$

粒子感受到力：$-m\dfrac{d\vec{v}_{DE}}{dt} \sim \vec{F}$

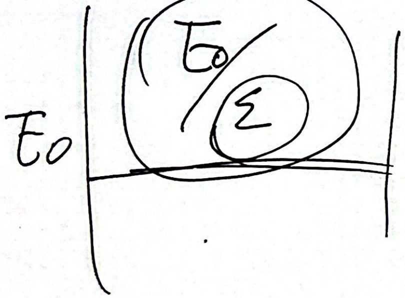
$$\cancel{\frac{d\vec{v}_c}{dt}} + \frac{d\vec{v}_{DE}}{dt} = \frac{q}{m}\left[\vec{E}(t) + \cancel{\vec{v}_c\times\vec{B}} + \vec{v}_{DE}\times\vec{B}\right].$$

$$\vec{v}_{Dp} = \frac{-\frac{m}{q}\frac{d\vec{v}_{DE}}{dt}\times\vec{B}}{B^2} = -\frac{m}{qB^4}\left(\frac{d\vec{E}}{dt}\times\vec{B}\right)\times\vec{B} = +\frac{m}{qB^2}\frac{d\vec{E}_\perp}{dt}$$

$$\vec{j}_p = nq\left(\frac{m_p}{qB^2}+\frac{m_e}{qB^2}\right)\frac{d\vec{E}}{dt} = \frac{n(m_p+m_e)}{B^2}\frac{d\vec{E}}{dt} = \frac{\rho}{B^2}\frac{d\vec{E}}{dt},\ \rho \text{ 质量密度}$$

$$\frac{\partial\vec{P}}{\partial t} = \vec{j}_p = \frac{\rho}{B^2}\frac{\partial\vec{E}}{\partial t} \Rightarrow \vec{P} = \frac{\rho\vec{E}}{B^2},\quad \vec{D} = \varepsilon_0\vec{E}+\vec{P} = \left(\varepsilon_0\vec{E}+\frac{\rho}{B^2}\vec{E}\right) = \varepsilon_0\vec{E}\left(1+\frac{\rho}{B^2\varepsilon_0}\right) = \varepsilon\varepsilon_0\vec{E}$$

介电常数 $\varepsilon = 1 + \frac{\rho}{B^2\varepsilon_0} = 1 + \frac{\rho\mu_0}{B^2\mu_0\varepsilon_0}$, $\frac{1}{\mu_0\varepsilon_0} = c^2 \Rightarrow \varepsilon = 1 + \frac{c^2}{\left(\frac{B^2}{\mu_0\rho}\right)} = 1 + \frac{c^2}{v_A^2}$



缓变电场无法穿入等离子体内部

$v_A << c \Rightarrow \varepsilon >> 1$

3.2 MHD 适用条件

单流体：大尺度、慢时标 ⇒ 可以忽略电子、质子差异，准中性

$L \gg r_{cp}$. $T \gg \tau_{cp}, \tau_{wp}$

$L \gg \lambda_{pe}$ $T \gg \tau \sim \omega_{pe}^{-1}$

流体力学：成团，碰撞足够频繁.

磁流体：库仑碰撞、磁场 → 约束/等效碰撞、异常碰撞：波粒作用

3.3 MHD 方程组

电子—质子 (e, p) 等离子体，$\alpha = e, p$.

$$\begin{cases} \dfrac{\partial \rho_\alpha}{\partial t} + \nabla\cdot(\rho_\alpha \vec{v}_\alpha) = 0 \\[2ex] \rho_\alpha \dfrac{d\vec{v}_\alpha}{dt} = -\nabla p_\alpha + \rho_\alpha \vec{g} + \rho_{q\alpha}(\vec{E} + \vec{v}_\alpha \times \vec{B}) + \vec{f}_{\alpha\beta} \\[2ex] \nabla\cdot\vec{E} = \rho_q/\varepsilon_0 \\ \nabla\cdot\vec{B} = 0 \\ \nabla\times\vec{E} = -\partial\vec{B}/\partial t \\ \nabla\times\vec{B} = \mu_0\vec{j} + \mu_0\varepsilon_0 \dfrac{\partial\vec{E}}{\partial t} \end{cases}$$

, $p_\alpha = n_\alpha k_B T_\alpha$

$$\begin{cases} \rho_q = \sum_\alpha n_\alpha q_\alpha \\ \vec{j} = \sum_\alpha n_\alpha q_\alpha \vec{v}_\alpha \end{cases}$$

ρ_q/ε_0 $-\dfrac{\partial \rho_q}{\partial t}$

$$\Rightarrow 0 = \cancel{\mu_0}\,\nabla\cdot\vec{j} + \mu_0\varepsilon_0 \frac{\partial \nabla\cdot\vec{E}}{\partial t}.$$

$$\Rightarrow \frac{\partial}{\partial t}\left(\nabla\cdot\vec{E} - \rho_q/\varepsilon_0\right) = 0$$

$$\frac{\partial(\rho_e+\rho_p)}{\partial t} + \nabla\cdot(\rho_e \vec{v}_e + \rho_p \vec{v}_p) = 0 \Rightarrow \boxed{\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho \vec{v}) = 0}$$

$\boxed{\vec{v} \sim \vec{v}_p}$

① 质量守恒

$\rho = \rho_e + \rho_p \approx \rho_p = n_p m_p = n m_p, \; n_p = n_e = n$

$$\frac{\partial n}{\partial t} + \nabla\cdot(n\vec{v}) = 0$$

动量方程.

② $\rho \frac{d\vec{v}}{dt} = -\nabla(p_e + p_p) + \rho\vec{g} + \sum_\alpha \cancel{\rho_\alpha}^{\,n q_\alpha} \cancel{(\vec{E} +)} \vec{v}_\alpha \times \vec{B} \longrightarrow \vec{j} \times \vec{B}$

$p = p_e + p_p = n k_B T_e + n k_B T_p = \boxed{2 n k_B T}$

$$\boxed{\rho \frac{d\vec{v}}{dt} = -\nabla p + \vec{j} \times \vec{B} + \rho \vec{g}}$$

$\vec{j} \times \vec{B} = 0$

无力场

Force-free field

$\boxed{\text{电荷守恒方程:}}$ $\frac{\partial \rho_q}{\partial t} + \nabla\cdot\vec{j} = 0$

③ 安培定律:

$\boxed{\mu_0 \varepsilon_0 = 1/c^2}$

$$\frac{B}{L} \Big/ \frac{1}{c^2}\frac{E}{T} \sim \frac{B}{E}\frac{T}{L} c^2 = \frac{c^2}{V^2} \gg 1$$

$$\frac{E}{L} \simeq \frac{B}{T} \Rightarrow E/B \sim L/T = V$$

$\nabla \times \vec{B} = \mu_0 \vec{j}$

无旋，势场

$\boxed{\nabla \times \vec{B} = 0}$

$$\vec{j}\times\vec{B} = \frac{\nabla\times\vec{B}}{\mu_0}\times\vec{B} = \frac{1}{\mu_0}\left(\vec{B}\cdot\nabla\vec{B} - \frac{\nabla B^2}{2}\right) = -\nabla\frac{B^2}{2\mu_0} + \frac{\vec{B}\cdot\nabla\vec{B}}{\mu_0}$$

$\frac{\varepsilon_0 E^2}{2}$, $\boxed{\frac{B^2}{2\mu_0}}$

$$\boxed{\rho\frac{d\vec{v}}{dt} = -\nabla\Big(\underbrace{p}_{\text{热压}} + \underbrace{\frac{B^2}{2\mu_0}}_{\text{磁压}}\Big) + \frac{\vec{B}\cdot\nabla\vec{B}}{\mu_0} + \rho\vec{g}}$$

$-\nabla p$: $P \to p$ 热运动的平均效果（热压梯度力）

$-\nabla p_B = -\nabla\frac{B^2}{2\mu_0}$ $B \to B$.（磁压梯度力）→ 磁压力

$\frac{\vec{B}\cdot\nabla\vec{B}}{\mu_0}$ $-\frac{\vec{R}_c}{R_c^2} \sim \frac{\vec{B}\cdot\nabla\vec{B}}{B^2}$ 磁张力.

㊀ 欧姆定律 $\boxed{\vec{j} = \sigma\vec{E},\ \vec{E} = \vec{j}/\eta}$

$$\rho_p\frac{d\vec{v}_p}{dt} = -\nabla p_p + en(\vec{E} + \vec{v}_p\times\vec{B}) + \vec{f}_{ep}$$

电子运动方程（广义欧姆定律）：

$$ne\vec{E} = -\nabla p_e - en(\vec{v}_e\times\vec{B}) + \vec{f}_{pe} - \rho_e\frac{d\vec{v}_e}{dt}$$

流体元坐标系： $\vec{j}' = \sigma \vec{E}'$. 以$\vec{v}$运动的参考系

实验室坐标系： $\cancel{\vec{j} = \sigma(\vec{E} + \vec{v}\times\vec{B})}$

$$\boxed{\vec{j} = \sigma(\vec{E} + \vec{v}\times\vec{B})}$$

$$\boxed{\vec{E} + \vec{v}\times\vec{B} = \vec{j}/\sigma}$$

$$\vec{E} = -\vec{v}\times\vec{B} + \eta\vec{j}, \quad \eta = \frac{1}{\sigma}$$

$\boxed{\vec{E} = -\vec{v}\times\vec{B}}$. $\vec{E} + \vec{v}\times\vec{B} = 0$, 理想 MHD. $(\sigma \to \infty)$ $(\eta = 0)$.

磁感应方程 $\boxed{\frac{\partial\vec{B}}{\partial t} = -\nabla\times\vec{E}}$ $\Rightarrow$ $\frac{\partial(\nabla\cdot\vec{B})}{\partial t} = 0$

$\nabla\cdot\vec{E} = \rho_q/\varepsilon_0$ ✓

$\nabla\cdot\vec{B} = 0$

$$\frac{\partial\vec{B}}{\partial t} = -\nabla\times(\underbrace{-\vec{v}\times\vec{B} + \eta\vec{j}}_{\vec{E}})$$

$$\frac{\partial\vec{B}}{\partial t} = \nabla\times(\vec{v}\times\vec{B}) - \nabla\times(\eta\nabla\times\vec{B}/\mu_0)$$

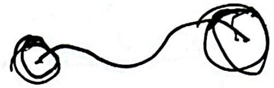
$$= \nabla\times(\vec{v}\times\vec{B}) - \frac{\eta}{\mu_0}\nabla\times(\nabla\times\vec{B})$$

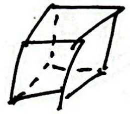
$$= \nabla\times(\vec{v}\times\vec{B}) + \frac{\eta}{\mu_0}\nabla^2\vec{B}$$

$$\boxed{\eta_m = \eta/\mu_0 = \frac{1}{\sigma\mu_0}}$$

$$\boxed{\frac{\partial \vec{B}}{\partial t} = \nabla \times (\vec{v} \times \vec{B}) + \eta_m \nabla^2 \vec{B}} \Leftarrow \left(\frac{\partial \vec{B}}{\partial t} = -\nabla \times \vec{E}, \quad \vec{E} = -\vec{v} \times \vec{B} + \eta \vec{j} \right)$$

MHD方程组：





$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\vec{v}) = 0$$

$$\frac{d\rho}{dt} + \cancel{\vec{v}\cdot\nabla\rho}\ \rho\nabla\cdot\vec{v} = 0 \quad , \left(\frac{d}{dt} = \frac{\partial}{\partial t} + \vec{v}\cdot\nabla\right).$$

$$\rho\frac{d\vec{v}}{dt} = -\nabla p + \vec{j}\times\vec{B} + \rho\vec{g}.$$

$$p = 2nk_BT$$

$$\vec{j} = \sum_\alpha n_\alpha q_\alpha \vec{v}_\alpha \times \vec{B}$$

$$\nabla\times\vec{B} = \mu_0\vec{j}$$

$$\frac{\partial\vec{B}}{\partial t} = -\nabla\times\vec{E} = \nabla\times(\vec{v}\times\vec{B}) + \eta_m\nabla^2\vec{B}$$

$$\vec{E} + \vec{v}\times\vec{B} = \vec{j}/\sigma$$

$$\nabla\times(\nabla\times\vec{B}) = -\nabla^2\vec{B}$$

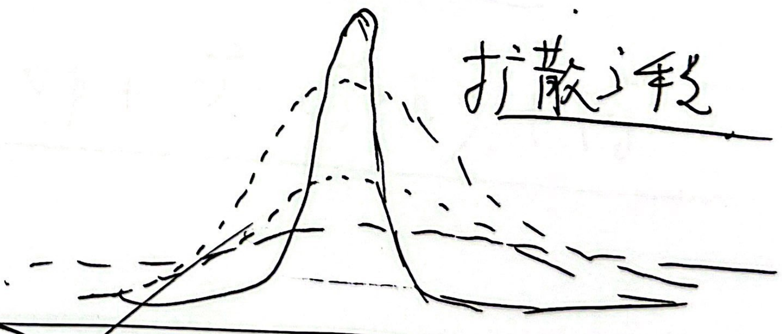
$$\frac{\partial}{\partial t}\left(\frac{p}{\gamma-1}\right) + \nabla\cdot\left(\frac{p\vec{v}}{\gamma-1}\right) + p\nabla\cdot\vec{v} = j^2/\sigma$$

$$\boxed{p = c\rho^\gamma} \qquad \eta_m = \frac{1}{\sigma\mu_0}$$

$$\Leftarrow \frac{d}{dt}(p/\rho^\gamma) = 0$$

$$\boxed{\frac{\partial T}{\partial t} \propto \eta_m \nabla^2 T}$$

扩散方程



$$\boxed{\vec{j}\cdot\vec{E}}$$

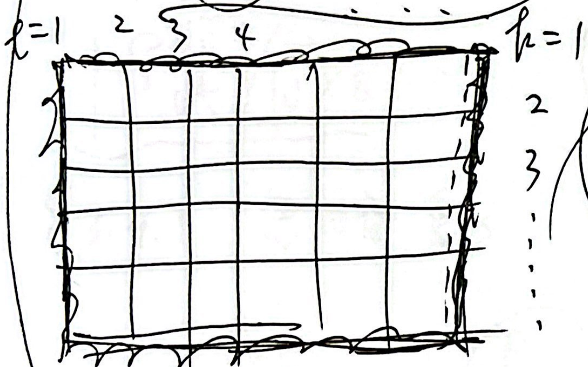
$\rho, \vec{v}, \vec{B}, p$

8个参量, 8个方程

$$\nabla\cdot\vec{B} = 0, \quad \nabla\cdot\vec{E} = \rho_q/\varepsilon_0$$

$$\frac{\partial \rho}{\partial t} \sim \frac{\rho_{i+1}^{t_i,m} - \rho_{i+1}^{t_i,m}}{t_{i+1} - t_{i-1}} = \cdots \quad \frac{\partial \rho}{\partial x} \sim \frac{\rho_t^{l+1,m} - \rho_t^{l-1,m}}{x_{l+1} - x_{l-1}}$$

由 $t_{i-1} \to t_{i+1}$



离散化

1° 初条件. $t_0 \to t_1 \to t_2 \to t_3 \to t_4$

2° 边条件.

$$v_x \left(v_x \frac{\partial}{\partial x} + v_y \frac{\partial}{\partial y} + v_z \frac{\partial}{\partial z}\right) v_x$$
$$+$$
$$v_y \left(v_x \frac{\partial}{\partial x} + v_y \frac{\partial}{\partial y} + v_z \frac{\partial}{\partial z}\right) v_y$$
$$+$$
$$v_z \left(v_x \frac{\partial}{\partial x} + v_y \frac{\partial}{\partial y} + v_z \frac{\partial}{\partial z}\right) v_z.$$

$$\rho_{i+1} = \rho_i + \Delta t * (\cdots\cdots)$$

能量密度分项: 动能 $\frac{\rho v^2}{2}$, 磁能 $\frac{B^2}{2\mu_0}$, 内能 $\frac{p}{\gamma - 1}$, 势能 $\frac{\rho G M_\odot}{r}$

$$\rho \vec{v} \cdot \frac{d\vec{v}}{dt} = \frac{\partial}{\partial t}\left(\frac{\rho v^2}{2}\right) + \nabla \cdot \left(\frac{\rho v^2}{2} \vec{v}\right) \quad \checkmark \quad = -\vec{v} \cdot \nabla p + \vec{v} \cdot (\vec{j} \times \vec{B}) + \rho \vec{g} \cdot \vec{v}$$

$$\rho \vec{v} \cdot \left(\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v}\right) = \rho \frac{\partial}{\partial t}\left(\frac{v^2}{2}\right) + \rho \vec{v} \cdot (\vec{v} \cdot \nabla \vec{v})$$

$$\vec{v} \cdot \left(v_x \frac{\partial}{\partial x} + v_y \frac{\partial}{\partial y} + v_z \frac{\partial}{\partial z}\right) \vec{v}$$

3.4 磁静平衡态 Magnetohydrostatics : MHS

$$0 = -\nabla p + \vec{j}\times\vec{B} + \rho\vec{g} \qquad \rho = n m_p$$

$$\left|\rho\frac{d\vec{v}}{dt}\right| \Big/ |\vec{j}\times\vec{B}| \sim \rho\frac{V}{T} \Big/ \frac{B^2}{\mu_0 L} \sim \frac{LV}{T} \Big/ \frac{B^2}{\mu_0\rho} = \frac{V^2}{V_A^2} \ll 1. \quad v \ll v_A = \frac{B}{\sqrt{\mu_0\rho}}$$

§3.4.1 忽略重力. $-\nabla p + \vec{j}\times\vec{B} = 0 = -\nabla(p + p_B) + \frac{\vec{B}\cdot\nabla\vec{B}}{\mu_0} = 0$

$$\frac{p/L}{\frac{B^2}{\mu_0 L}} \sim \frac{p}{\frac{B^2}{2\mu_0}} = \frac{\beta}{2},$$

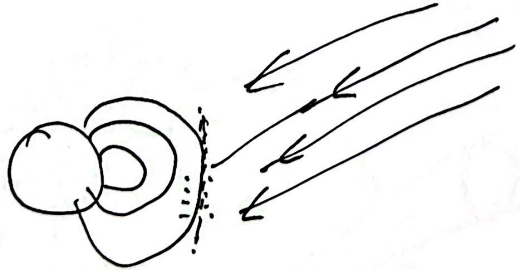
定义 $\beta = \frac{p}{p_B}$, 等离子体热压与磁压之比.

无量纲参数.

$\beta \gg 1$, $\beta \sim 1$, $\beta \ll 1$

$p \gg p_B$ $\quad p \sim p_B$ $\quad p \ll p_B$

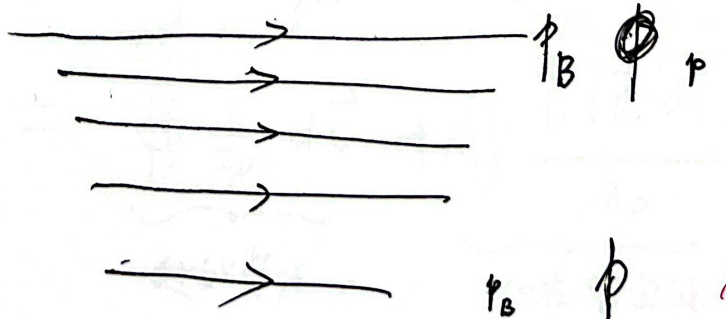
热 — , 温 $\quad$ 冷等离子体

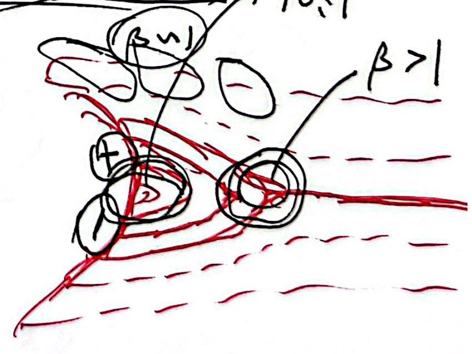


$$-\nabla p + \vec{j}\times\vec{B} = -\nabla(p + p_B) + \frac{\vec{B}\cdot\nabla\vec{B}}{\mu_0} = 0$$

㊀ 垂直向力线. $-\nabla(p + p_B) = 0$ 总压平衡

低β.





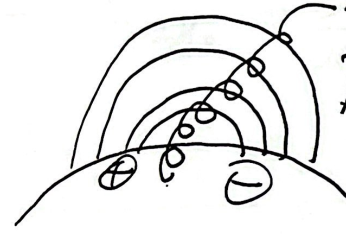
3.4.2 无力场与势场

$$\boxed{0 = \vec{j} \times B}$$ （低β等离子体）

$$\sum_i \vec{j}_i \times \vec{B}_j = 0 \qquad -\nabla p_B + \frac{\vec{B}\cdot\nabla\vec{B}}{\mu_0} = 0$$

$$100 - 100 = 0$$
$$99 - 101 \sim 0$$

磁绳 $\vec{B}_r$ 环向、轴向电流（磁场）



$$\mu\vec{j} = \nabla \times \vec{B}$$

无力场：$\vec{j} \parallel \vec{B}$

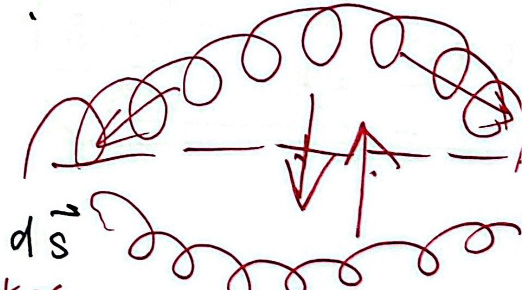
$$\nabla \times \vec{B} = \alpha \vec{B}$$

$$\Rightarrow (\vec{B}\cdot\nabla)\alpha = 0$$

证明：对于∀无力场，有

沿磁力线，α是一个常数。

势场：$\boxed{\nabla \times \vec{B} = 0}$

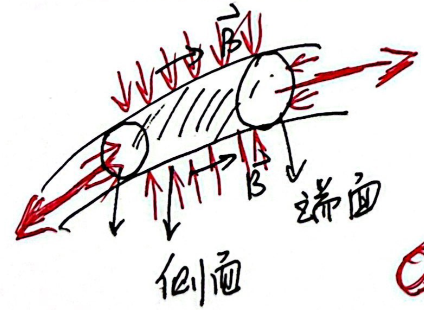


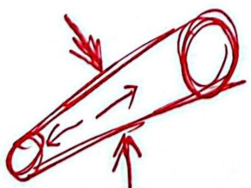
3.4.3 磁压力与磁张力

$$\vec{f} = -\nabla p_B + \frac{\vec{B}\cdot\nabla\vec{B}}{\mu_0} = -\nabla\cdot\left(p_B \overleftrightarrow{I} - \frac{\vec{B}\vec{B}}{\mu_0}\right)$$

$$\oiiint \vec{f}\, d\tau = -\oiint \left(\frac{B^2}{2\mu_0}\overleftrightarrow{I} - \frac{\vec{B}\vec{B}}{\mu_0}\right)\cdot d\vec{s}$$

$$= -\underbrace{\overbrace{\oiint \frac{B^2}{2\mu_0} d\vec{s}}^{\text{磁压力}}}_{\text{处处受力}} + \underbrace{\overbrace{\oiint \frac{\vec{B}(\vec{B}\cdot d\vec{s})}{\mu_0}}^{\text{磁张力}}}_{\text{仅在端面处受力}}$$





记: $\vec{f} = \vec{j} \times \vec{B} = -\nabla \frac{B^2}{2\mu_0} + \frac{\vec{B}\cdot\nabla\vec{B}}{\mu_0} = -\nabla_\perp \frac{B^2}{2\mu_0} - \nabla_{\parallel} \frac{B^2}{2\mu_0} + \frac{\vec{B}\cdot\nabla(B\vec{s})}{\mu_0}$

$\vec{B}\cdot\nabla \equiv B\frac{\partial}{\partial s}$

$= -\nabla_\perp \frac{B^2}{2\mu_0} - \frac{\partial(B^2/2\mu_0)}{\partial s}\hat{s} + \left(\frac{B\frac{\partial B}{\partial s}}{\mu_0}\right)\hat{s} + \frac{B^2\frac{\partial \hat{s}}{\partial s}}{\mu_0} = \frac{B\frac{\partial(B\hat{s})}{\partial s}}{\mu_0}$

$\frac{\partial \hat{s}}{\partial s}$

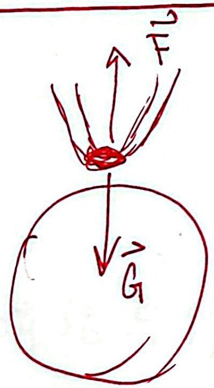
$\vec{s}$

$B\vec{s}$

$= -\nabla_\perp \frac{B^2}{2\mu_0} - \frac{B^2}{\mu_0}\frac{\vec{R}_c}{R_c^2}$

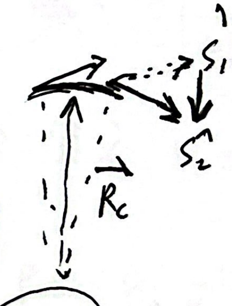
$+ \frac{B^2}{\mu_0}(- \quad = \frac{\vec{B}\cdot\nabla\vec{B}}{\mu_0}$

平行分量抵消



$\frac{\vec{B}\cdot\nabla\vec{B}}{B^2\mu_0} =$

$\vec{B}\cdot\nabla(B\hat{s})$

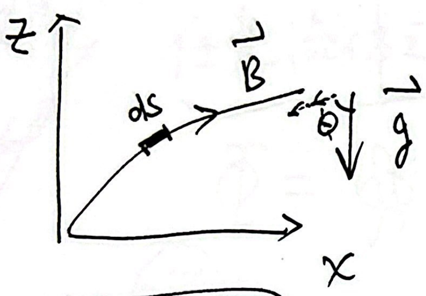


$\frac{\partial \hat{s}}{\partial s} = -\frac{\vec{R}_c}{R_c^2}$

$\frac{\vec{B}\cdot\nabla\vec{B}}{B^2} = -\frac{\vec{R}_c}{R_c^2}$ (2.35)

3.4.5

3.4.5 重力分层大气的标高 (MHS)



$$0 = -\nabla_{\parallel} p + \rho \vec{g}_{\parallel} = -\frac{dp}{ds} - \rho g \cos\theta$$

$$ds = \frac{dz}{\cos\theta}$$

$$\Rightarrow -\frac{dp}{dz} = \rho g$$

$$p = 2 n k_B T = 2 \rho k_B T / m_p$$

$$\Rightarrow -\frac{2k_BT}{m_p}\frac{d\rho}{dz} = \rho g$$

$$\Rightarrow \frac{1}{\rho}\frac{d\rho}{\cancel{dz}} = -\frac{m_p g}{2k_B T} dz$$

$\Rightarrow$ 定义 $H = \frac{2k_BT}{m_p g}$.

$$\rho = \rho_0 e^{-z/H}$$

$$p = p_0 e^{-z/H}.$$

标高 $\boxed{H = \frac{2k_BT}{m_p g}}$

光球附近: $H \approx 5000\,K \Rightarrow H \sim 300\,km$ $\sim 4\times10^{-4} R_\odot$

$1 R_\odot = 6.95\times10^5\,km$.

日冕附近: $H \sim 1\,MK$, $\Rightarrow H \sim 60\,Mm$ $6\times10^4\,km$ $\sim 0.1 R_\odot$

3.5 磁冻结与扩散.

$$\boxed{\frac{\partial\vec{B}}{\partial t} = -\nabla\times\vec{E} = \nabla\times(\vec{v}\times\vec{B}) + \eta_m \nabla^2\vec{B}}$$

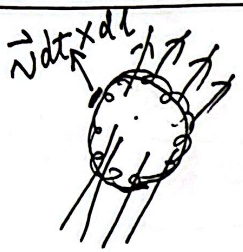
理想 (ideal) MHD: $\sigma\to\infty$, $\eta_m = \frac{1}{\sigma\mu_0} \to 0$. $\underbrace{\frac{\partial\vec{B}}{\partial t} = \nabla\times(\vec{v}\times\vec{B})}$.

耗散 (resistive) MHD: σ 有限. ✓.

$$R_m = \frac{vB/L_B}{\eta_m B/L_B^2} \sim \frac{L_B V}{\eta_m}$$

估算: $L_B \sim 0.01 R_\odot \sim 10^6 \sim 10^7\,m$, $V \sim 1\,km/s \sim 10^3$, $\eta_m \sim 1\,m^2/s$ $\Big\}$ $R_m \sim 10^{10}$

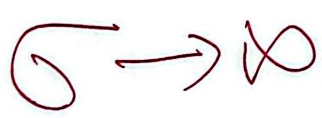
3.5.1 Alfven 磁冻结定理.



任意流体之回路中所包含的 Φ 不随时间变化.

$$\Phi = \oiint \vec{B}\cdot d\vec{s}\ , \quad \frac{d\Phi}{dt} = 0$$

$$\oiint \frac{\partial \vec{B}}{\partial t}\cdot d\vec{s} + \oint \vec{B}\cdot(\vec{v} \times d\vec{l}) \cancel{/dt}$$

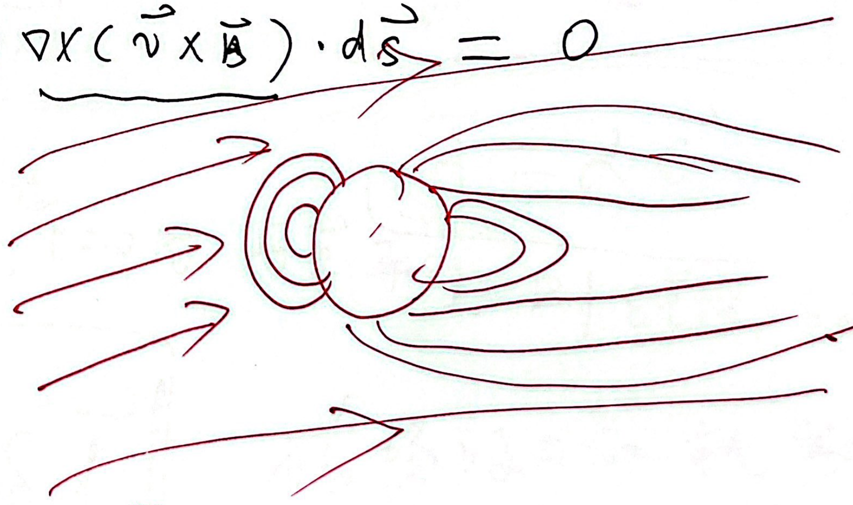


$$\oiint \frac{\partial \vec{B}}{\partial t}\cdot d\vec{s} + \oint \vec{B}\times\vec{v}\cdot d\vec{l} = \oiint \frac{\partial \vec{B}}{\partial t} d\vec{s} - \oint (\vec{v}\times\vec{B})\cdot d\vec{l}$$

$$= \oiint \frac{\partial \vec{B}}{\partial t}\cdot d\vec{s} - \oiint \nabla\times(\vec{v}\times\vec{B})\cdot d\vec{s} = 0$$

证明:

$$\boxed{\frac{d(\vec{B}/\rho)}{dt} = \frac{\vec{B}\cdot\nabla\vec{v}}{\rho}}$$



$$\frac{1}{\rho}\frac{d\vec{B}}{dt} - \frac{\vec{B}}{\rho^2}\frac{d\rho}{dt} = \frac{1}{\rho}\left(\frac{\partial \vec{B}}{\partial t} + \vec{v}\cdot\nabla\vec{B}\right)$$

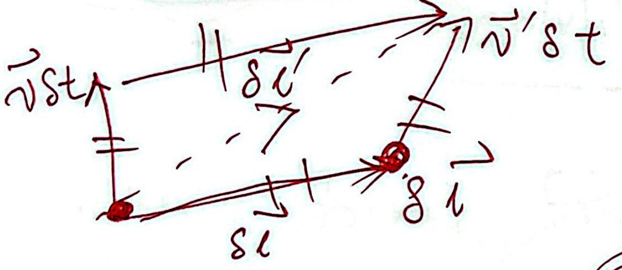
$$- \frac{\vec{B}}{\rho^2}\left(\frac{\partial \rho}{\partial t} + \vec{v}\cdot\nabla\rho\right)$$

$$= \frac{1}{\rho}\left(\nabla\times(\vec{v}\times\vec{B}) + \vec{v}\cdot\nabla\vec{B}\right) + \frac{\vec{B}}{\rho}\nabla\cdot\vec{v} = 0.$$

$$\frac{\partial \rho}{\partial t} + \vec{v}\cdot\nabla\rho + \rho\nabla\cdot\vec{v} = 0$$

$\vec v_1$ $\vec v_2$ 线元 $\delta\vec l$

证明: $\boxed{\frac{d(\delta\vec l)}{dt} = \delta\vec l\cdot\nabla\vec v}$



$$\delta\vec l + \vec v' dt = \delta\vec l' + \vec v\delta t$$

$$\frac{(\delta\vec l' - \delta\vec l)}{dt} = \vec v' - \vec v \equiv \delta\vec l\cdot\nabla\vec v$$

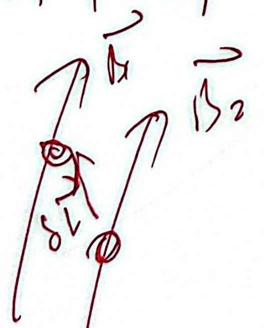
$$\boxed{\frac{d}{dt}(\vec B/\rho) = \frac{\vec B}{\rho}\cdot\nabla\vec v \qquad \frac{d}{dt}(\delta\vec l) = \delta\vec l\cdot\nabla\vec v}$$

$\vec B/\rho$ 与 $\delta\vec l$ 满足同样的方程

$$\boxed{\frac{d\vec A}{dt} = \vec A\cdot\nabla\vec v \qquad \frac{d\vec C}{dt} = \vec C\cdot\nabla\vec v}$$

$$\vec A = \alpha\vec C$$

如果两个流体元构成的线元 $(\delta\vec l)$，在 $t=0$ 时，与 $\vec B$ 平行，则 $\delta\vec l(t)$ 与 $\vec B$ 始终平行

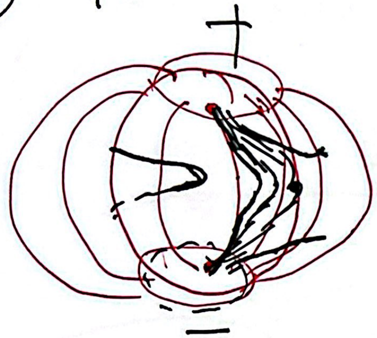


$$\boxed{|\vec B/\rho| = \alpha|\delta\vec l|}$$

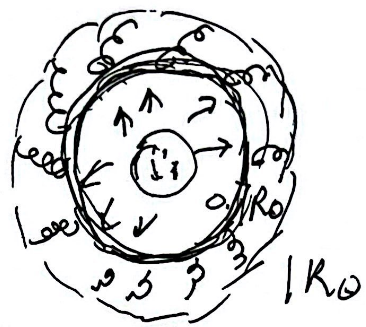
磁场增大的机制

发电机理论

① 太阳发电机 (Solar Dynamo)



较差自转. 0.7 $R_\odot$

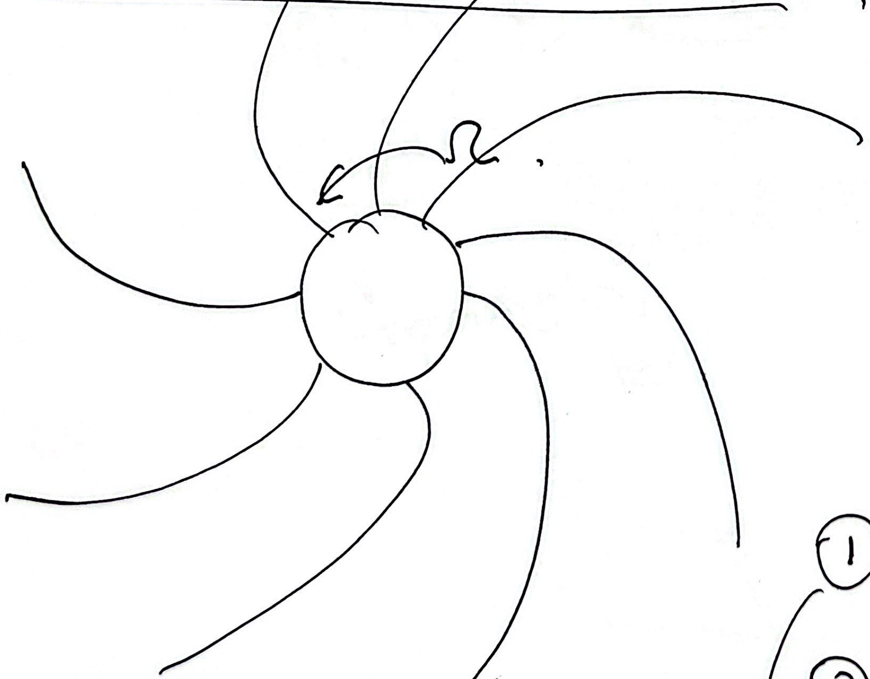


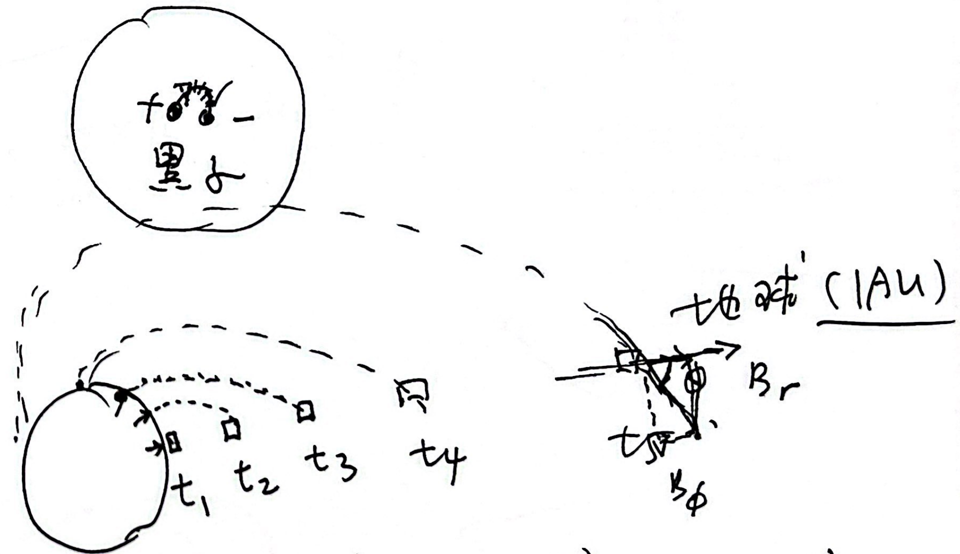
tachline layer

$-\nabla \frac{B^2}{2\mu_0} \sim \rho \frac{d\vec{v}}{dt}$

日震学

② Parker's Spiral Field





① 共转参考系, $\vec{v}' = v_r \vec{e}_r - \Omega r \vec{e}_\phi$, $\vec{v} /\!/ \vec{B}$

② 实验室参考系. $\vec{v} = v_r \vec{e}_r \sim 400 km/s$

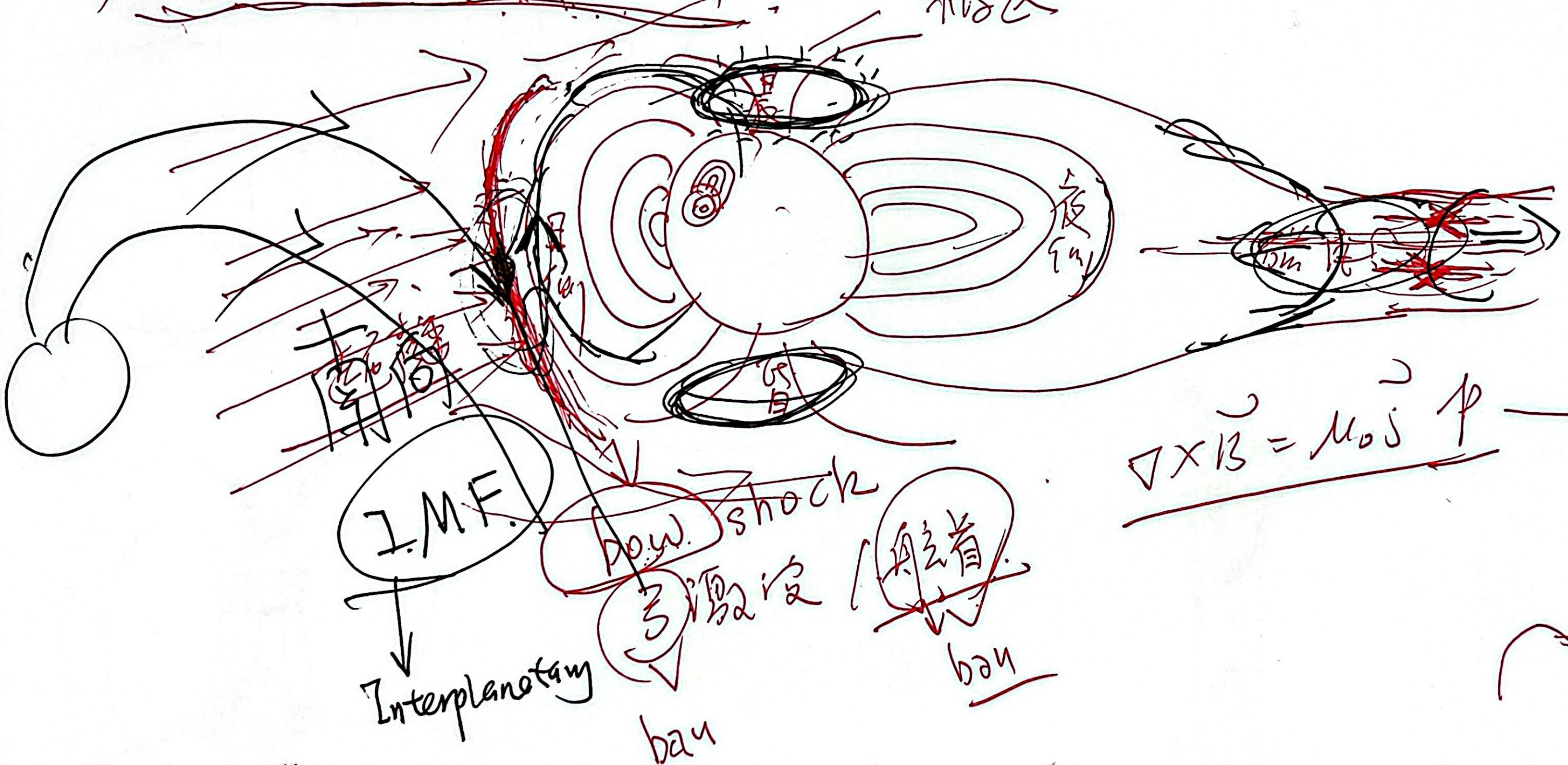
$\theta \sim 45^\circ \Leftarrow \frac{4\times10^5}{2.7\times10^{-6}\times1.5\times10^{11}}$

$1 = \operatorname{ctg}\theta \Rightarrow \frac{v_r}{v_\phi} = \frac{B_r}{B_\phi} \Rightarrow \frac{B_r}{B_\phi} = \frac{v_r}{\Omega r} = \frac{400 km/s}{2.7\times10^{-6} \times 1.5\times10^{11} m}$

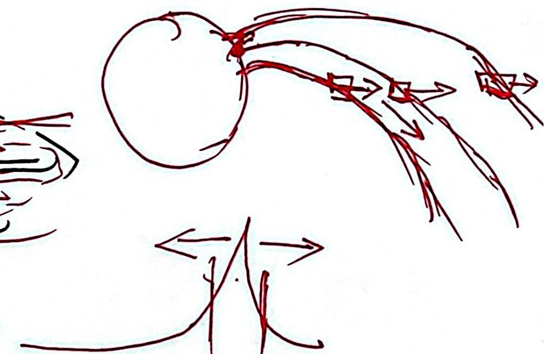
1) Solar Dynamo. 2) Parker's spiral field, 3) CIR: 共转相互作用区

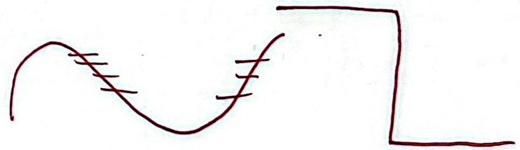
Stream-stream IR.

4) 太阳风—磁层相互作用



$\nabla\times\vec{B} = \mu_0\vec{j}$ ↑



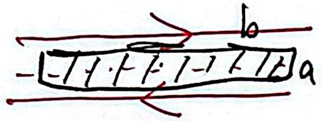


3.5.2 磁扩散

$$\frac{\partial \vec{B}}{\partial t} = \cancel{\nabla\times(\vec{v}\times\vec{B})} + \eta_m \nabla^2\vec{B}$$

L_B 越的小，η_m ↑

扩散时标：$\frac{B}{\tau_B} \sim \eta_m \frac{B}{L_B^2} \Rightarrow \tau_B = \frac{L_B^2}{\eta_m} \sim \frac{(10^6)^2}{1} s \sim 10^{12} s$

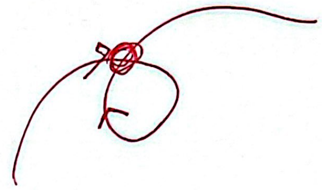


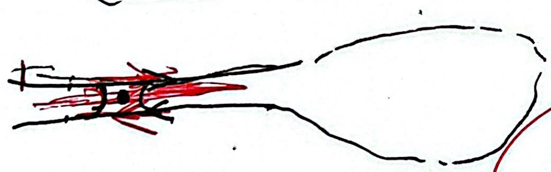
$S = ab$

$\oint \vec{B}\cdot d\vec{l} = \mu_0 I$ 　 $\vec{j} = I/S$

ηj^2 焓：磁场湮灭.

短路.



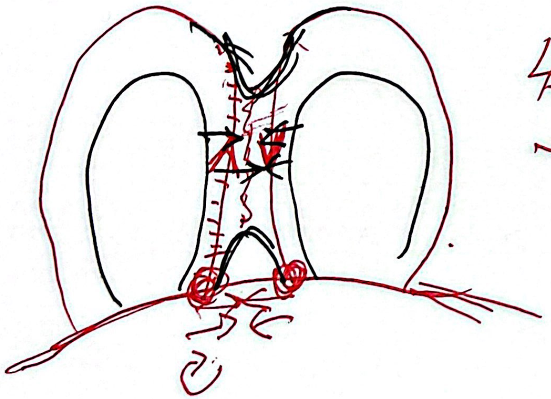


why

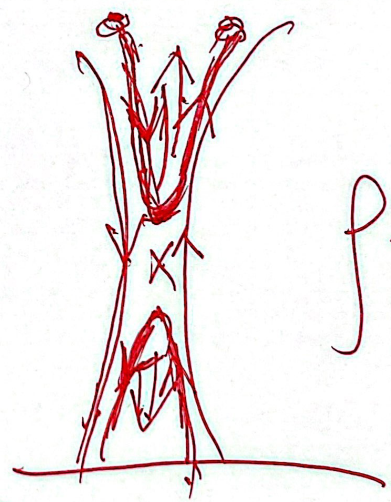
爆发式？

不稳定性

instability



磁张力



弹弓效应

$$\rho \frac{d\vec{v}}{dt} = -\nabla(p + p_B) + \frac{\vec{B}\cdot\nabla\vec{B}}{\mu_0}$$

能量释放：如何？哪儿去？

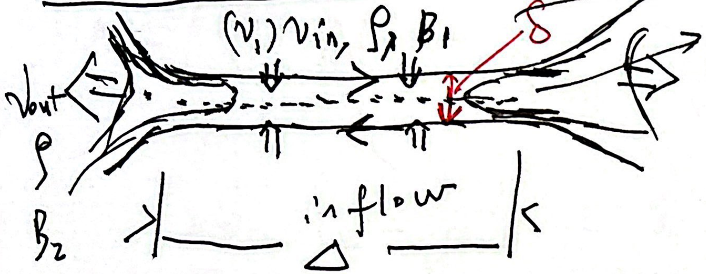
加热 ηj^2

重联电场、高能粒子（电子质子）？

整体加速

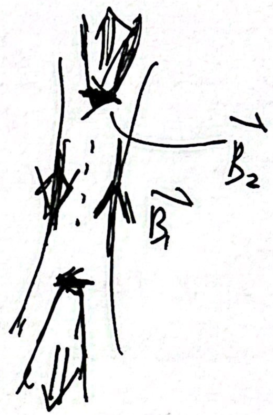
Bulk Acceleration

Sweet-Parker 模式.（慢，稳态重联模式）



电流片/中性片

current/neutral sheet



入流区，出流区，耗散区（扩散区）：冻结条件被破坏（$\eta_m \neq 0$）

$\underbrace{\text{耗散区（扩散区）}}_{\text{重联区}}$

重联率：reconnection rate $M = \dfrac{v_{in}}{v_{out}} = \dfrac{v_1}{v_2}$?

决定 v_1 的是：电阻效应. $\tau_B = L_B^2/\eta_m$, $\eta_m = \dfrac{1}{\mu_0 \sigma}$, $\boxed{\tau_B = L_B^2 \mu_0 \sigma}$

① 质量守恒：$\cancel{\rho}\, v_1 \cancel{(t)} \cdot \Delta \cdot \cancel{H} = \cancel{\rho} \cdot v_2 \cancel{(t)}\, \delta \cdot \cancel{H} \Rightarrow M = \dfrac{v_1}{v_2} = \dfrac{\delta}{\Delta}$

② $B_1 \delta H = B_2 \Delta H \Rightarrow \dfrac{B_1}{B_2} = \dfrac{\Delta}{\delta} \Rightarrow B_2 = \dfrac{\delta}{\Delta} B_1$

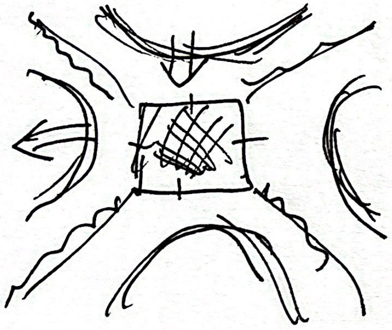
③ v_1, v_2：$v_1 = \dfrac{\delta (=L_B)}{L_B^2 \mu_0 \sigma} = \dfrac{1}{\mu_0 \sigma \delta}$

$$\frac{\rho v_2^2}{2} \simeq \underbrace{\frac{\Delta}{2} \cdot \vec{j} \times \vec{B}}_{\text{功}} \simeq \frac{\Delta}{2} \cdot \frac{\nabla \times \vec{B_1}}{\mu_0} \times \vec{B_2} = \frac{\Delta}{2} \frac{B_1 B_2}{\mu_0 \delta} = \frac{\Delta B_1^2 \delta / \Delta}{2 \mu_0 \delta}$$

$\therefore v_2^2 = \dfrac{B_1^2}{\mu_0 \rho} = v_{A1}^2$, $\underline{v_2 = v_{A1}}$.

$$M = \frac{v_1}{v_2} \quad \left(\frac{\Delta}{\mu_0 \sigma \delta v_{A1}} = \frac{\delta^2}{\Delta} \Rightarrow \delta = \sqrt{\frac{\Delta}{\mu_0 \sigma v_{A1}}}, \quad M = \frac{\delta}{\Delta} = \frac{1}{\sqrt{\mu_0 \sigma \Delta v_{A1}}} \right.$$

设 $\eta_m = \frac{1}{\mu_0 \sigma} = 1$, $\Delta \sim 10\,km$ $\quad v_A \sim 1000\,km/s$ $\quad M \sim 10^{-5}$

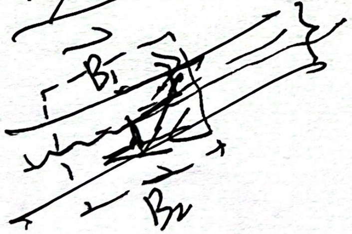


Petschek 模式

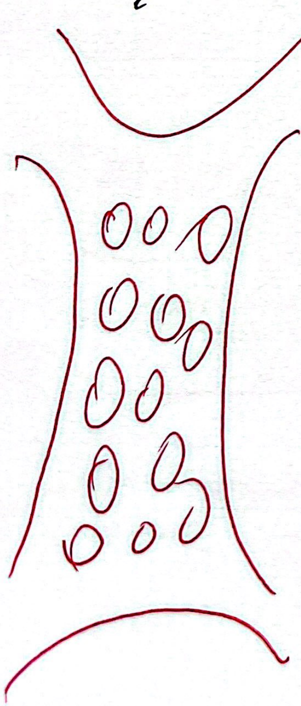
① 驱动重联

② 自发重联 Spontaneous Rec.

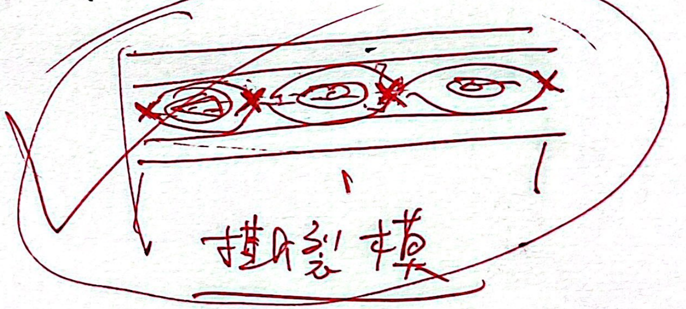
~~MHD~~ → 数值重联.



PIC 重联.



撕裂模



~~磁岛~~

3.7 理想 MHD 线性波模

3.7.1 流体力学声波与真空电磁波

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \nabla\cdot(\rho \vec{v}) = 0 \\ \rho \dfrac{d\vec{v}}{dt} = \rho\left(\dfrac{\partial \vec{v}}{\partial t} + \vec{v}\cdot\nabla\vec{v}\right) = -\nabla p \\ p = c\rho^{\gamma}, \quad c = p_0/\rho_0^{\gamma} \end{cases}$$

$$p = c(\rho_0+\rho')^{\gamma} = c\rho_0^{\gamma} + \gamma c \rho_0^{\gamma-1}\rho' + \cdots, \quad p = p_0 + p'$$

$$\boxed{p' = \gamma c \rho_0^{\gamma-1}\rho'} = \gamma \frac{p_0}{\rho_0}\rho'$$

线性化(小扰动假设)，$\rho = \rho_0 + \rho'$，$\vec{v}$ $(\vec{v}_0 = 0)$，$p = p_0 + p'$

$\rho' \ll \rho_0$ $\quad p' \ll p_0$.

$(\rho_0 + \rho')\vec{v}$

线性化处理：

$\rho'\cdot\vec{v} = 0$ $\quad \vec{v}\cdot\vec{v} = 0$

$$\Rightarrow \begin{cases} \dfrac{\partial \rho'}{\partial t} + \rho_0 \nabla\cdot\vec{v} = 0 \\ \rho_0 \dfrac{\partial \vec{v}}{\partial t} = -\nabla p' = -\gamma\dfrac{p_0}{\rho_0}\nabla\rho' \end{cases} \Rightarrow \rho_0 \frac{\partial \nabla\cdot\vec{v}}{\partial t} = -\gamma \frac{p_0}{\rho}\nabla^2\rho'$$

$$\Uparrow$$

$$\frac{\partial^2 \rho'}{\partial t^2} = c_s^2 \nabla^2 \rho'$$

$$\boxed{c_s^2 = \gamma p_0/\rho_0}$$

$\rho' \sim \bar{\rho}\sin(\omega t + kx) \Rightarrow -\omega^2\rho' = -c_s^2 k^2 \rho' \Rightarrow \boxed{\omega = k c_s}$

$\omega = \frac{2\pi}{T},\ T = \frac{2\pi}{\omega},\ k = \frac{2\pi}{\lambda},\ \lambda = 2\pi/k.$

扰动量 $Q' = \bar{Q}\exp(i\vec{k}\cdot\vec{r} - i\omega t) = \sum_{\omega, k} \bar{Q}\left(\cos(\vec{k}\cdot\vec{r} - \omega t) + i\sin(\vec{k}\cdot\vec{r} - \omega t)\right)$

$(\rho', \vec{v})$ $\boxed{\frac{\partial}{\partial t}Q' = -i\omega Q',\ \nabla\cdot Q' = i\vec{k}Q'}$ $\frac{\partial}{\partial x}Q' = ik_x Q'.$

FT分析

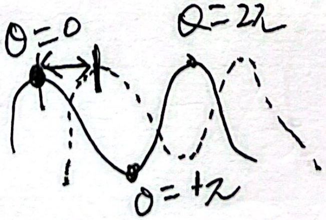
$\Rightarrow \begin{cases} -i\omega\rho' + i\vec{k}\cdot\vec{v}\rho_0 = 0 \\ -i\omega\rho_0\vec{v} = -c_s^2 i\vec{k}\rho' \Rightarrow -i\omega\rho_0 \vec{k}\cdot\vec{v} = -c_s^2 k^2 i\rho' \end{cases}$

声波

① 纵波 longitudinal Mode $\Rightarrow \omega^2 = k^2 c_s^2.$ $\omega = k c_s$ $\boxed{\omega/k} = c_s = \frac{\partial p}{\partial \rho}$

$\vec{k} \parallel \vec{v}$

② $\frac{\rho'}{\rho_0} = \frac{\vec{k}\cdot\vec{v}}{\omega} = \frac{kv}{\omega} = \frac{v}{c_s}$



$\vec{c_s} = c_s \vec{k}/k.$

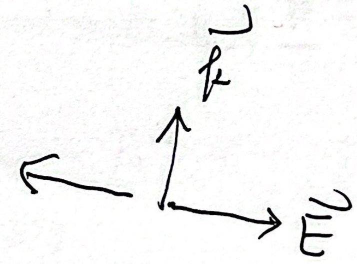
$$\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \qquad \vec{B} = \vec{B}_0 + \vec{B}', \quad \vec{E} = \vec{E}_0 + \vec{E}'$$

$$\nabla\times\vec{B} = \cancel{\mu_0\vec{j}} + \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t} \qquad (\vec{k}\perp\vec{E})$$

$$\Rightarrow \begin{cases} \vec{k}\times\vec{E} = +\omega\vec{B} \\ \vec{k}\times\vec{B} = -\frac{1}{c^2}\omega\vec{E} \end{cases} \Rightarrow \vec{k}\times(\vec{k}\times\vec{E}) = -\frac{\omega^2}{c^2}\vec{E}$$

$$-k^2\vec{E}$$

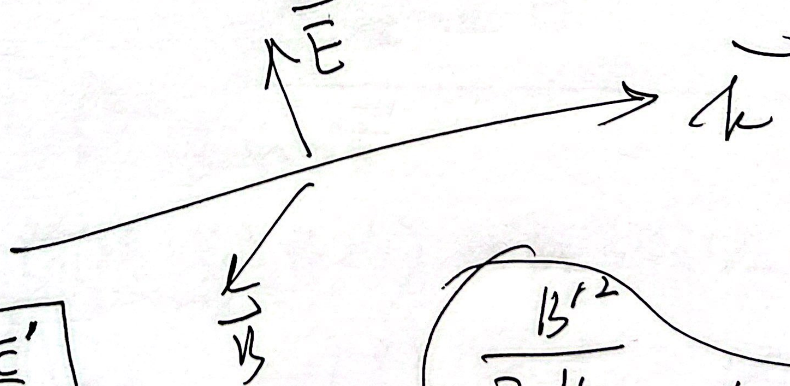
$$\Rightarrow \omega = kc$$



$\omega/k = c, \quad \frac{d\omega}{dk} = c,$

$\vec{E}\parallel\vec{k}$：纵波

$\vec{E}\perp\vec{k}$：横波



$kE = \omega B$

$E = Bc, \quad \boxed{B' = \frac{E'}{c}}$

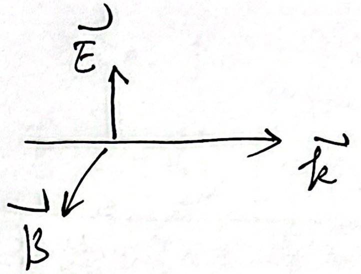
$$\frac{B'^2}{2\mu_0} = \frac{E'^2\varepsilon_0\mu_0}{2\mu_0} = \frac{\varepsilon_0E'^2}{2}$$

$\frac{\varepsilon_0E'^2}{2}$

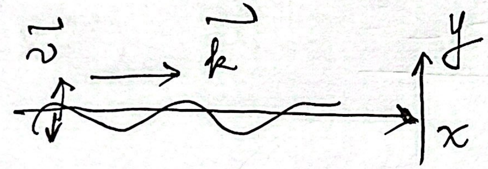
$\rho \frac{d\vec{v}}{dt} = -\nabla p + \vec{j} \times \vec{B} + \rho \vec{g}$

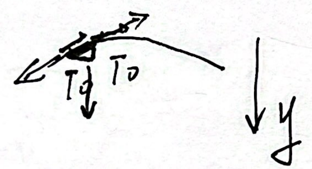
① $\omega = k C_s$, $C_s^2 = \frac{\gamma p_0}{\rho_0}$ 纵波 $\vec{v} \Leftrightarrow \vec{k}$

② $\left.\begin{cases} \nabla \times \vec{B} = \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \\ \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \end{cases}\right\} \Rightarrow \omega = kc$ 横波



③ 弦振动. $\vec{v} \perp \vec{k}$.





$$\boxed{\frac{\partial^2 y}{\partial t^2} = \frac{T_0}{\rho} \frac{\partial^2 y}{\partial x^2}}, \quad V = \sqrt{\frac{T_0}{\rho}}$$

$$T_0 \sin\alpha_1 - T_0 \sin\alpha_2 = T_0 \operatorname{tg}\alpha_1 - T_0 \operatorname{tg}\alpha_2 = T_0 (y'_{x+dx} - y'_x)$$
$$= \rho\, dx \frac{d^2 y}{dt^2}.$$

3.7.2 $\boxed{\text{说明}}$.

① 寻找小扰动(线性)对应的简谐波解, $A = \sum_{\omega, k} \vec{A}_{\omega, k}\, e^{i\vec{k}\cdot\vec{r} - i\omega t}$

背景均匀, $\frac{\partial}{\partial t} \sim -i\omega$, $\nabla \sim i\vec{k}$

线性化后的方程组：一阶(线性的,齐次)代数方程组.

$$\begin{pmatrix} & & \\ & & \\ & & \end{pmatrix} \begin{pmatrix} \rho \\ v \\ B \end{pmatrix} = 0 \qquad \vec{A} \begin{pmatrix} \rho \\ v \\ B \end{pmatrix} = 0 \Rightarrow \det(\vec{A}) = 0$$

3-7-3 剪切Alfvén波

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \nabla\cdot(\rho\vec{v}) = 0 \\ \rho\dfrac{d\vec{v}}{dt} = -\nabla p + \vec{j}\times\vec{B} = -\nabla(p+p_B) + \dfrac{\vec{B}\cdot\nabla\vec{B}}{\mu_0} \\ \dfrac{\partial \vec{B}}{\partial t} = \nabla\times(\vec{v}\times\vec{B}), \quad \nabla\cdot\vec{B}=0 \\ p = C\rho^\gamma \end{cases}$$

$$\frac{d\vec{v}}{dt} = \frac{\partial \vec{v}}{\partial t} + \vec{v}\cdot\nabla\vec{v}$$

$$p_B = \frac{B^2}{2\mu_0} = \frac{(\vec{B}_0+\vec{B}')^2}{2\mu_0} = p_{B0} + \frac{2\vec{B}_0\cdot\vec{B}'}{2\mu_0} + \frac{(\vec{B}')^2}{2\mu_0} \Rightarrow -\nabla p_B \sim \nabla\frac{(\vec{B}_0\cdot\vec{B}')}{\mu_0}$$

① 小扰动假设

B_0. ρ_0. $v_0=0$. $\vec{E}_0=0$. p_0.

$\Leftarrow \vec{B}=\vec{B}_0+\vec{B}'$, $\rho=\rho_0+\rho'$, v, $\vec{E}$, $p=p_0+p'$

② 线性化忽略二阶项.

$$\begin{cases} \dfrac{\partial \rho'}{\partial t} + \rho_0\nabla\cdot\vec{v} = 0 \\ \rho_0\dfrac{\partial \vec{v}}{\partial t} = -c_s^2\nabla\rho' - \nabla\dfrac{\vec{B}_0\cdot\vec{B}'}{\mu_0} + \dfrac{\vec{B}_0\cdot\nabla\vec{B}'}{\mu_0} \\ \dfrac{\partial \vec{B}'}{\partial t} = \nabla\times(\vec{v}\times\vec{B}_0), \quad \vec{k}\cdot\vec{B}'=0 \end{cases}$$

③ $A = \sum_{\omega,\vec{k}} A e^{i\vec{k}\cdot\vec{r}-i\omega t}$

$$\Rightarrow \begin{cases} \dfrac{\partial}{\partial t} \to -i\omega \\ \nabla \to i\vec{k} \end{cases}$$

$$\Rightarrow \begin{cases} -i\omega\rho' + i\vec{k}\cdot\vec{v}\rho_0 = 0 \\ -i\omega\rho_0\vec{v} = -ic_s^2\vec{k}\rho' - i\vec{k}\dfrac{\vec{B}_0\cdot\vec{B}'}{\mu_0} + \dfrac{i\vec{B}_0\cdot\vec{k}}{\mu_0}\vec{B}' \\ -i\omega\vec{B}' = i\vec{k}\times(\vec{v}\times\vec{B}_0), \quad \vec{k}\cdot\vec{B}'=0. \end{cases}$$

消去 ρ'，$\vec{B_0}\cdot\vec{B'}$，保留 $\vec{k}\cdot\vec{v}$.

$$\frac{\rho'}{\rho_0}=\frac{\vec{k}\cdot\vec{v}}{\omega}$$

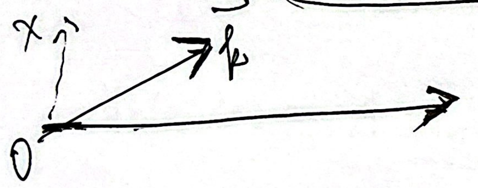
$$-i\omega\vec{B'}=i\left(\vec{k}\cdot\vec{B_0}\,\vec{v}-\vec{k}\cdot\vec{v}\,\vec{B_0}\right)$$

$\cdot\vec{B_0}$ $\Rightarrow$ $(\vec{k}\cdot\vec{B_0})(\vec{v}\cdot\vec{B_0})=0$

$\vec{v}\cdot\vec{B_0}=0$.

① $\vec{k}\cdot\vec{v}=0$，② $\vec{k}\cdot\vec{v}\neq 0$

$\rho'=0$，$\vec{k}\cdot\vec{B'}=0$，$\boxed{\vec{B_0}\cdot\vec{B'}=0}$



$\boxed{\text{动量方程}\cdot\vec{B_0}\Rightarrow\vec{v}\cdot\vec{B_0}=0}$

△ $\rho'=0\longrightarrow$ 不可压，$\vec{v}\parallel\vec{B'}$，$\vec{v}\perp(\vec{k},\vec{B_0})$ $\vec{B'}\perp(\vec{k},\vec{B_0})$

$$\Rightarrow -i\omega\rho_0\vec{v}_m=\frac{i\vec{B_0}\cdot\vec{k}}{\mu_0}\vec{B'}\Rightarrow -i\omega\vec{B'}=i\vec{k}\cdot\vec{B_0}\,\frac{i\vec{B_0}\cdot\vec{k}}{\mu_0}\vec{B'}\Big/(-i\omega\rho_0)$$

$$-i\omega\vec{B'}=i\vec{k}\cdot\vec{B_0}\,\vec{v}$$

$$\Downarrow$$

$$\Rightarrow\omega^2\vec{B'}=\frac{(\vec{k}\cdot\vec{B_0})^2}{\rho_0\mu_0}\vec{B'}$$

$$\Rightarrow\omega^2=\frac{(\vec{k}\cdot\vec{B_0})^2}{\mu_0\rho_0}，\text{令}\ \vec{v}_A=\frac{\vec{B_0}}{\sqrt{\mu_0\rho_0}}$$

$$\Rightarrow\omega^2=(\vec{k}\cdot\vec{v}_A)^2$$

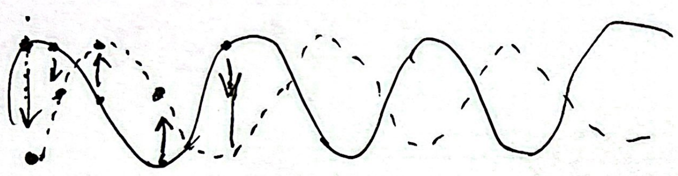
剪切 Alfvén 波

Shear Alfvén wave

色散关系

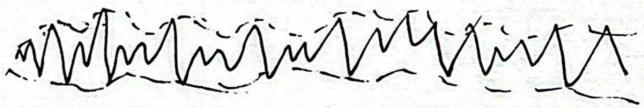
$$\omega=\vec{k}\cdot\vec{v}_A=k_{\parallel}v_A=kv_A\cos\theta$$

(2) $\vec{v}_p$. $\vec{v}_g$



$\underline{\omega = k c_s} \longrightarrow \boxed{v_p = c_s = v_g}$

$\underline{\omega = kc} \longrightarrow \boxed{v_p = c = v_g}$



$$\sin\alpha + \sin\beta = 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$

$$y_1(x,t) = \underline{A}\,\underline{\sin(k_1x - \omega_1 t)} \qquad y_2(x,t) = \underline{A}\,\underline{\sin(k_2x - \omega_2 t)}$$

$$y = y_1 + y_2 \Rightarrow 2A\sin\Big(\underbrace{\frac{(k_1+k_2)x}{2}}_{\bar{k}} - \underbrace{\frac{(\omega_1+\omega_2)}{2}}_{\bar{\omega}}t\Big)\cos\Big(\underbrace{\frac{k_1-k_2}{2}}_{\frac{\Delta k}{2}}x - \underbrace{\frac{(\omega_1-\omega_2)}{2}}_{\frac{\Delta\omega}{2}}t\Big).$$

$$\frac{\Delta\omega}{\Delta k} = v_g.$$

$$\vec{v}_p = \frac{\omega}{k}\frac{\vec{k}}{k}, \quad \vec{v}_g = \frac{d\omega}{d\vec{k}} = \frac{\partial\omega}{\partial k_x}\hat{e}_x + \frac{\partial\omega}{\partial k_y}\hat{e}_y + \frac{\partial\omega}{\partial k_z}\hat{e}_z$$

$$v_g = \frac{d\omega}{dk} = \frac{d(kv_p)}{dk} = v_p + k\frac{dv_p}{dk} = v_p - \lambda\frac{dv_p}{d\lambda},$$

$$\underline{v_p > v_g} \qquad v_p < v_g.$$

磁张力 $\longrightarrow$ 不可压 剪切 Alfvén 波，$\omega = k v_A \cos\theta$，$(\vec{v}, \vec{B}') \perp (\vec{k}, \vec{B}_0)$.

$$-\omega \rho_0 \vec{v} = \frac{\vec{B}_0 \cdot \vec{k}}{\mu_0} \vec{B}' \Longrightarrow -k \cdot v_A \vec{v} = \frac{\vec{B}_0 \cdot \vec{k}}{\rho_0 \mu_0} \vec{B}'$$

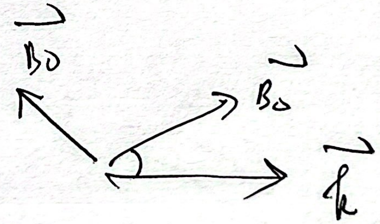
$$\left| \frac{\vec{v}}{\vec{B}'} \right| = \frac{\cancel{k} B_0 \cos\theta}{\sqrt{\mu_0 \rho_0}\ \cancel{k B_0 \cos\theta}}\, B_0 = \frac{v_A}{B_0}$$

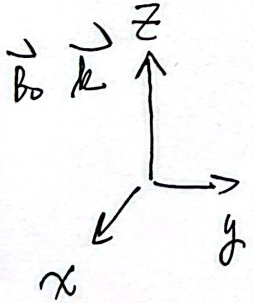
$$\left| \frac{v}{v_A} \right| = \left| \frac{B}{B_0} \right| .$$

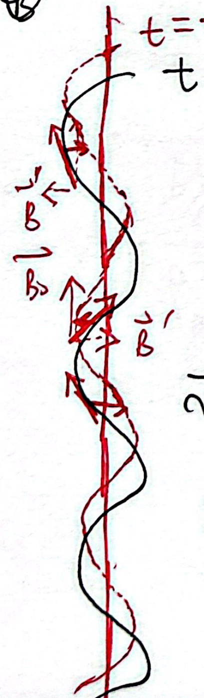
$$v_A = B_0 / \sqrt{\mu_0 \rho_0} .$$

$$\boxed{\frac{\rho'}{\rho_0} = \frac{v}{c_s}} .$$

$$\vec{v} = \pm \frac{v_A}{B_0} \vec{B}'$$?

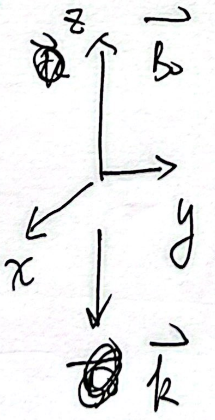


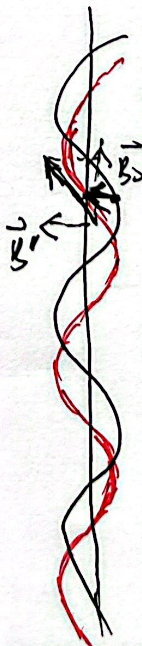




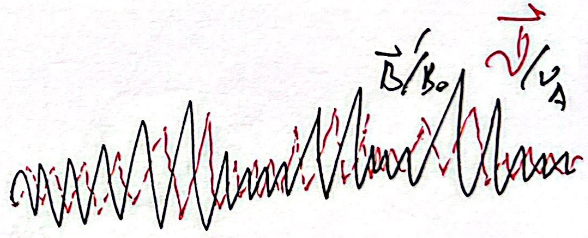
$\vec{v} \parallel \vec{B}'$

$\vec{v} \parallel \vec{B}'$





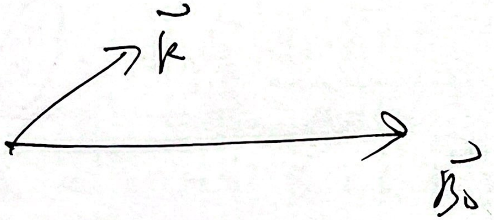
$\vec{v} \parallel \vec{B}'$



$$\vec{v} = -\frac{v_A}{B_0} \vec{B}' \ \mathrm{sign}(\vec{k} \cdot \vec{B}_0)$$

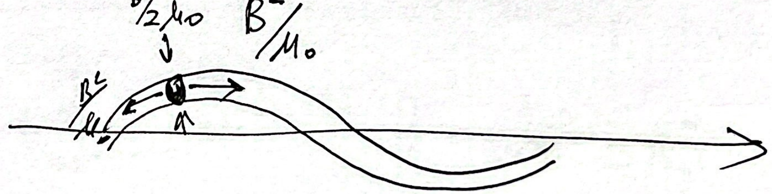
相速度 $\underline{\underline{v_p}} = \frac{\omega}{k}\frac{\vec{k}}{k} = v_A|\cos\theta|\,\vec{k}/k$, $\quad \underline{\underline{\vec{v}_g}} = \frac{\partial\omega}{\partial k_{\parallel}}\hat{e}_z = \underline{v_A \hat{e}_z \,\mathrm{sign}(\vec{k}\cdot\vec{B}_0)}$

① $\rho' = 0$, $(\vec{B}, \vec{v}) \perp (\vec{k}, \vec{B}_0)$, $\left|\frac{\vec{v}}{v_A}\right| = \left|\frac{\vec{B}'}{B_0}\right|$, $\underline{\omega = k_{\parallel} v_A = k v_A \cos\theta}$.

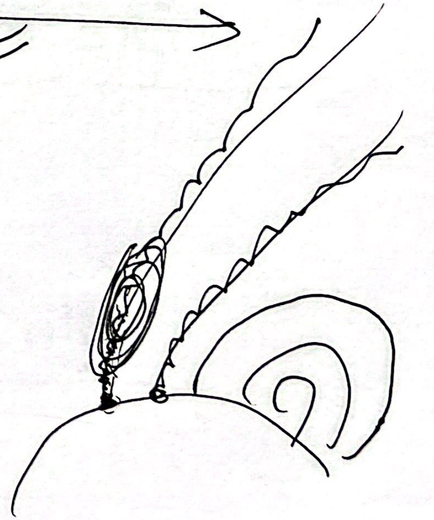


(张力项: $\vec{B}\cdot\nabla\vec{B}/\mu_0$ $\quad (\vec{B}_0\cdot\vec{k})\vec{B}'/\mu_0$

$$\underline{v = \sqrt{T/\rho}}$$



$$= \sqrt{\frac{B^2/\mu_0 \cdot \cancel{S}}{\rho_0 \cancel{S}}} = \frac{B_0}{\sqrt{\mu_0 \rho_0}} = \boxed{v_A},$$



3.7.4 可压缩 MHD波

① $\vec{k}\cdot\vec{v}=0\ (\rho'=0)$ ； ② $\vec{k}\cdot\vec{v}\neq 0\ (\rho'\neq 0)$ ，？ $\rho\dfrac{d\vec{v}}{dt}=-\nabla p_B+\dfrac{\vec{B}\cdot\nabla\vec{B}}{\mu_0}$

⇓ 剪切 Alfvén 波

$c_s^2=0$（冷等离子体）

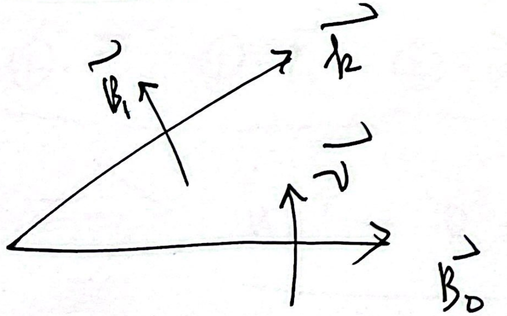
$$\begin{cases} ①\ -i\omega\rho_0\vec{v} = -i\vec{k}\dfrac{\vec{B}_0\cdot\vec{B}'}{\mu_0} + \dfrac{i\vec{B}_0\cdot\vec{k}}{\mu_0}\vec{B}_1 \cancel{-i\vec{k}\rho_1 c_s^2} \\ ②\ -i\omega\vec{B}_1 = -i\vec{B}_0(\vec{k}\cdot\vec{v}) + i(\vec{B}_0\cdot\vec{k})\vec{v} \end{cases}$$

$$\frac{\rho'}{\rho_0}=\frac{\vec{k}\cdot\vec{v}}{\omega}$$

$$\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\vec{v})=0$$

$$-i\omega\rho'+i\vec{k}\rho_0\cdot\vec{v}=0$$

共面定理：证毕。（考虑热运动时亦成立）



$\vec{k}\perp\vec{B}_1$ $\vec{v}$？ $\vec{v}\perp\vec{B}_0$

$$①\cdot\vec{k} \Rightarrow -i\omega\rho_0\vec{k}\cdot\vec{v} = -ik^2\frac{\vec{B}_0\cdot\vec{B}_1'}{\mu_0} = -i\omega^2\rho'$$

$$②\cdot\vec{B}_0 \Rightarrow -i\omega\vec{B}_1'\cdot\vec{B}_0 = -iB_0^2\,\vec{k}\cdot\vec{v}$$

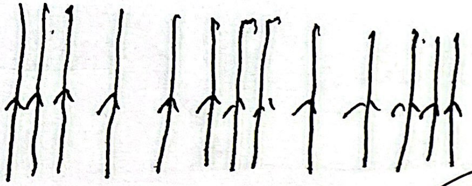
$$\Rightarrow \omega\rho_0\vec{k}\cdot\vec{v} = k^2\frac{B_0^2}{\mu_0\omega}\vec{k}\cdot\vec{v} \Rightarrow \omega^2=k^2v_A^2$$

$$\omega = kv_A$$

Compressible Alfven wave：可压缩 Alfvén（阿尔芬、阿尔文）波

恢复力（restoring force）：磁压力（$\vec{k}$）$-\nabla p_B$ + 磁张力

$\begin{cases}\text{平行：不可压剪切 Alfvén 波} \\ \text{垂直：仅磁压力起作用！}\end{cases}$



(垂直传播) → 纵波

疏密变化 $\vec{k}\perp\vec{B}_0$ ($\vec{B}' \parallel B_0$) → 压缩作用: $\vec{k}\cdot\vec{v}\neq 0$

声致振荡 $\vec{k}\parallel\vec{v}$

$$\vec{E} = -\vec{v}\times\vec{B}_0$$

② $\vec{k}\cdot\vec{v}\neq 0$ $(\rho'\neq 0)$, $c_s^2\neq 0$

声波 $(-\nabla p)$ + 可压 Alfvén 波 $(\vec{j}\times\vec{B})$ 的耦合: 磁声波 Magnetosonic Wave

<1> 共振定理 ✓ <2> 推导 $w = w(\vec{k})$

$① \cdot \vec{k}$, $① \cdot \vec{B}_0$, $② \cdot \vec{k} = 0$, $② \cdot \vec{B}_0$ $\quad \vec{k}\cdot\vec{v}$, $\vec{v}\cdot\vec{B}_0$, $\vec{B}'\cdot\vec{B}_0$

$$\Rightarrow w^4 - k^2 w^3 (c_s^2 + v_A^2) + k^2 c_s^2 (\vec{k}\cdot\vec{v}_A)^2 = 0$$

$\sqrt{\Delta}$, $\Delta \geq 0$

$$v_p^2 = \left(\frac{w}{k}\right)^2 = \frac{(c_s^2+v_A^2) \pm \sqrt{(c_s^2+v_A^2)^2 - 4c_s^2 v_A^2 \cos^2\theta}}{2} \geq 0$$

+: 快磁声波 Fast Magnetosonic wave (Fast wave/mode)

−: 慢 ——

快, 慢 ? $-i\vec{k}\rho_1 c_s^2 - i\vec{k}\dfrac{\vec{B}_0\cdot\vec{B}'}{\mu_0}$, $\dfrac{\rho_1}{\rho_0} = \dfrac{\vec{k}\cdot\vec{v}}{w}$

$① \cdot \vec{k} \Rightarrow$

$$-iw\rho_0\vec{k}\cdot\vec{v} = -ik^2\frac{\vec{B}_0\cdot\vec{B}'}{\mu_0} - ik^2c_s^2\frac{\vec{k}\cdot\vec{v}}{w}\rho_0$$

$$-\cancel{i}\omega^2 \rho_1 = -\cancel{i} k^2 \frac{\vec{B_0}\cdot\vec{B'}}{\mu_0} + \cancel{i} k^2 c_s^2 \rho_1 \Rightarrow \rho_1(\omega^2 - k^2 c_s^2) = k^2 \frac{\vec{B_0}\cdot\vec{B'}}{\mu_0}$$

$$\boxed{\begin{cases} \omega^2 > k^2 c_s^2, & \text{同向受力（热压力与磁压力）} \\ \omega^2 < k^2 c_s^2, & \text{反} \end{cases}}$$

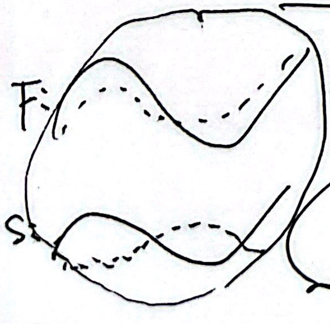
分析 F, S 波色散关系：

F $\begin{cases} \theta = 0^\circ \text{时}, & v_{pf\,min}^2 = \dfrac{c_s^2 + v_A^2 + \sqrt{(c_s^2 - v_A^2)^2}}{2} = \dfrac{c_s^2 + v_A^2 + |c_s^2 - v_A^2|}{2} = \begin{cases} c_s^2 & c_s^2 \geq v_A^2 \\ v_A^2 & c_s^2 < v_A^2 \end{cases} = \max(c_s^2, v_A^2) \\ \theta = 90^\circ \text{时}, & v_{pf\,max}^2 = c_s^2 + v_A^2 \end{cases}$

F: $\boxed{\max(c_s^2, v_A^2) \leq v_{pf}^2 \leq c_s^2 + v_A^2}$

S: $\theta = 0^\circ$ 时, $v_{ps\,max}^2 = \min(c_s^2, v_A^2)$

$\theta = 90^\circ$ 时. $v_{ps\,min}^2 = 0$, （不传播）



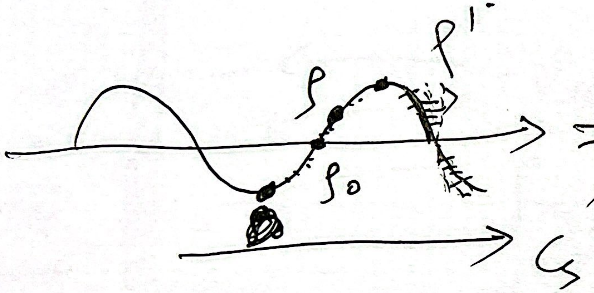
$$0 \leq v_{ps}^2 \leq \min(c_s^2, v_A^2) \leq \max(c_s^2, v_A^2) \leq v_{pf}^2 \leq c_s^2 + v_A^2$$

$\vec{v}_p = \frac{\omega}{k}\frac{\vec{k}}{k}$，$v_{p\parallel} = \frac{\omega}{k}\cos\theta$，$v_{p\perp} = \frac{\omega}{k}\sin\theta$

$\vec{v}_g$ 群速度：快波基本各向同性；慢波基本与 $\vec{B}_0$ 平行，Alfvén 波与 $\vec{B}_0$ 平行。

3.8 MHD 激波

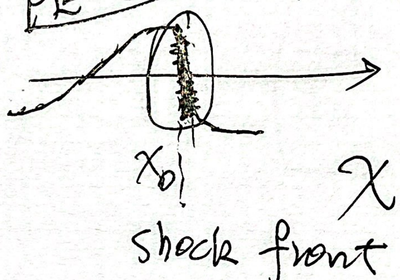
3.8.1 . 流体力学激波



$\Longrightarrow$

非线性效应

陡化 steepening



ρ' ① 梯度 $\nabla \rho \Big|_{x_0}$ ② 耗散效应 $\Longrightarrow$ shock

陡化效应：(一维声波)

$$\boxed{\frac{\rho'}{\rho_0} = \frac{v'}{c_{s0}}} \qquad c_{s0} = \gamma p_0 / \rho_0 .$$

$$c_s' = v + c_s = \frac{2}{\gamma-1}(c_s - c_{s0}) + c_s$$

$$= c_{s0}\left\{1 + \frac{\gamma+1}{\gamma-1}\left[\left(\frac{\rho}{\rho_0}\right)^{\frac{\gamma-1}{2}} - 1\right]\right\}.$$

压缩区 c_s' > 稀疏区 c_s' $\Longrightarrow$ 陡化.

$$\boxed{\frac{\delta\rho}{\rho} = \frac{\delta v}{c_s}} . \qquad c_s^2 = \gamma p / \rho$$

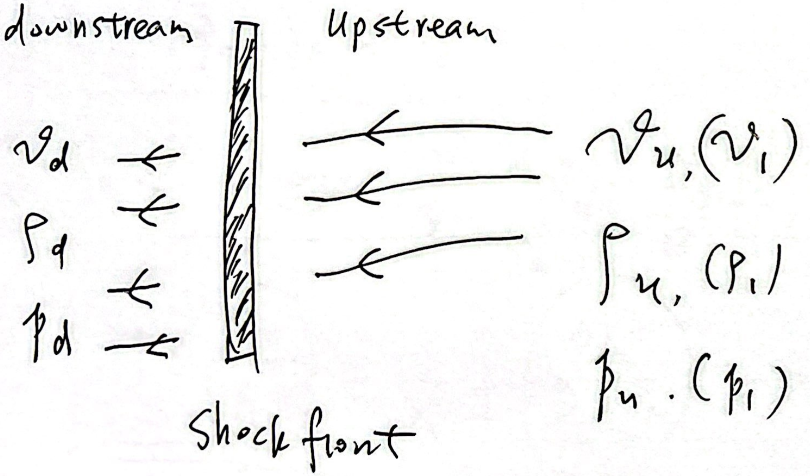
$$v = \int_{\rho_0}^{\rho} \frac{c_s}{\rho}\delta\rho \qquad \boxed{\rho = \rho_0 ,\ v = 0}$$

$$c_s^2 = \frac{\gamma p}{\rho} = c_{s0}^2\left(\frac{\rho}{\rho_0}\right)^{\gamma-1}, \quad c_{s0}^2 = \frac{\gamma p_0}{\rho_0}$$

$$\boxed{p = c\rho^{\gamma}} \quad \frac{p}{\rho} = c\rho^{\gamma-1} = \frac{\gamma p_0}{\rho_0}\left(\frac{\rho}{\rho_0}\right)^{\gamma-1}$$

$$v = \int_{\rho_0}^{\rho} c_{s0}\left(\frac{\rho}{\rho_0}\right)^{\frac{\gamma-1}{2}} \rho^{-1} d\rho = \frac{2}{\gamma-1}\frac{c_{s0}}{\rho_0^{\frac{\gamma-1}{2}}} \rho^{\frac{\gamma-1}{2}} \Big|_{\rho_0}^{\rho} = \boxed{\frac{2}{\gamma-1}(c_s - c_{s0})}$$

3.8.2 流体力学激波 RH 跳变关系 (jump conditions)



Shock frame

$$\frac{\rho_d}{\rho_u} \quad \frac{v_d}{v_u}, \quad \frac{p_d}{p_u}. \quad \frac{T_d}{T_u}$$

$$\begin{cases} \frac{\partial \rho}{\partial t} + \nabla(\rho\vec{v}) = 0 \\ \rho\left(\frac{\partial \vec{v}}{\partial t} + \vec{v}\cdot\nabla\vec{v}\right) = -\nabla p \Rightarrow \underline{\frac{\partial(\rho\vec{v})}{\partial t} + \nabla\cdot(\rho\vec{v}\vec{v}) + \nabla\cdot(p\vec{\underline{I}}) = 0} \\ p = c\rho^{\gamma} \\ \frac{\partial}{\partial t}\left[\rho\left(\varepsilon + \frac{v^2}{2}\right)\right] + \nabla\cdot\left[\rho\left(\varepsilon+\frac{v^2}{2}\right)\vec{v}\right] = -\nabla\cdot(p\vec{v}) \Rightarrow \rho\left(\varepsilon+\frac{v^2}{2}\right)v + \frac{p}{\rho}\rho v \Big|_{u-d} = 0 \end{cases}$$

$\rho\varepsilon v + pv = v\left(p + \frac{p}{\gamma - 1}\right) = \frac{\gamma p v}{\gamma-1}$

$$\Rightarrow \begin{cases} \rho\vec{v}\cdot\vec{n}\Big|_u = \rho\vec{v}\cdot\vec{n}\Big|_d \Rightarrow \underline{\rho v_n\Big|_{u-d} = 0} \\ \rho\underline{\vec{v}\vec{v}\cdot\vec{n}} + p\vec{n}\Big|_{u-d} = 0 \Rightarrow \underline{\rho v^2 + p}\Big|_{u-d} = 0 \end{cases}$$

$\rho\varepsilon = \frac{p}{\gamma-1}$

① $\rho_u v_u = \rho_d v_d$

② $\rho_u v_u^2 + p_u = \rho_d v_d^2 + p_d$

③ $\rho_u \frac{v_u^3}{2} + \frac{\gamma}{\gamma-1}\frac{p_u}{\rho_u} v_u \rho_u = \rho_d \frac{v_d^3}{2} + \frac{\gamma}{\gamma-1}\frac{p_d}{\rho_d} v_d \rho_d$

① 压缩比 $X = \frac{\rho_d}{\rho_u} = \frac{v_u}{v_d}$

② 马赫数 $M = \frac{v_u}{c_{su}}$, $c_{su}^2 = \frac{\gamma p_u}{\rho_u}$

③ p_d / p_u ,

$$\frac{p_u}{\rho_u v_u^2} = \frac{\gamma p_u}{\gamma \rho_u v_u^2} = \frac{1}{\gamma M^2}$$

②′ $1 + \frac{p_u}{\rho_u v_u^2} = \frac{1}{X} + \frac{p_d}{\rho_u v_u^2} \Rightarrow 1 + \frac{1}{\gamma M^2} = \frac{1}{X} + \frac{p_d}{p_u}\frac{1}{\gamma M^2}$

③′ $\frac{v_u^2}{2} + \frac{\gamma}{\gamma-1}\frac{p_u}{\rho_u} = \frac{v_d^2}{2} + \frac{\gamma}{\gamma-1}\frac{p_d}{\rho_d} \Rightarrow 1 + \frac{\gamma}{\gamma-1}\frac{2p_u}{\rho_u v_u^2} = \frac{v_d^2}{v_u^2} + \frac{\gamma}{\gamma-1}\frac{2p_d}{\rho_d v_u^2}$

$$1 + \frac{2\cancel{\gamma}}{\gamma-1}\frac{1}{\cancel{\gamma} M^2} = \frac{1}{X^2} + \frac{2\gamma}{\gamma-1}\frac{p_d}{p_u}\frac{p_u \rho_u}{\rho_d \rho_u v_u^2} = \frac{1}{\cancel{X^2}} + \frac{2\cancel{\gamma}}{\gamma-1}\frac{p_d}{p_u}\frac{1}{X}\frac{1}{\cancel{\gamma} M^2}$$

$$X^2\left(\frac{2}{M^2} + \gamma - 1\right) = 2\gamma\left[X\left(1 + \frac{1}{\gamma M^2}\right) - 1\right] + \gamma - 1 = 0$$

$$\Rightarrow X^2\left(2 + (\gamma-1)M^2\right) - 2\gamma\left(M^2 + \frac{1}{\gamma}\right)X + M^2(\gamma+1) = 0$$

$$\Delta = 4(M^2-1)^2$$

$(M>1)$

$$X = \frac{2\gamma M^2 + 2 \pm (2M^2 - 2)}{2(2 + (\gamma-1)M^2)} = \begin{cases} \text{“}+\text{”},\ X = \frac{\gamma M^2 + M^2}{2 + (\gamma-1)M^2} > 1 \\ \text{“}-\text{”},\ X = \frac{2\gamma M^2 + 4 - 2M^2}{4 + 2\gamma M^2 - 2M^2} = 1 \end{cases}$$

$M=1$, $X=1$; $M \to \infty$, $X \to \frac{\gamma+1}{\gamma-1}$ $\gamma = \frac{5}{3}$ $X \to 4$ ✓

$M>1$, $X>1$; $\frac{v_u}{c_{su}} \to \infty$ $\frac{v_u}{v_d} \to 4$

$$\frac{p_d}{p_u} = \frac{2\gamma M^2 - (\gamma-1)}{\gamma+1}.$$

说明 $\boxed{v_d < c_{s2}}$

下流是亚声速的？

$p_u = \rho_u R_u T_u$;

$$\frac{T_d}{T_u} = \frac{p_d}{\rho_d}\frac{\rho_u}{p_u} = \frac{p_d}{p_u}\cdot\frac{1}{X}$$

$$\boxed{\frac{\rho_u v_u^2}{p_u} = \gamma M^2}$$

$$\frac{v_d^2}{c_{s2}^2} = \frac{v_d^2}{v_u^2}\,\frac{v_u^2}{\gamma p_d/\rho_d} = \frac{1}{X^2}\,\frac{\rho_d v_u \rho_u p_u}{\gamma p_d \rho_u p_u} = \frac{1}{X}\,\frac{p_u}{p_d}\,M^2 = \frac{M^2}{X}\cdot\frac{\gamma+1}{[2\gamma M^2-(\gamma-1)]}.$$

激波：压缩、加热(压)，减速 slow down & heat up

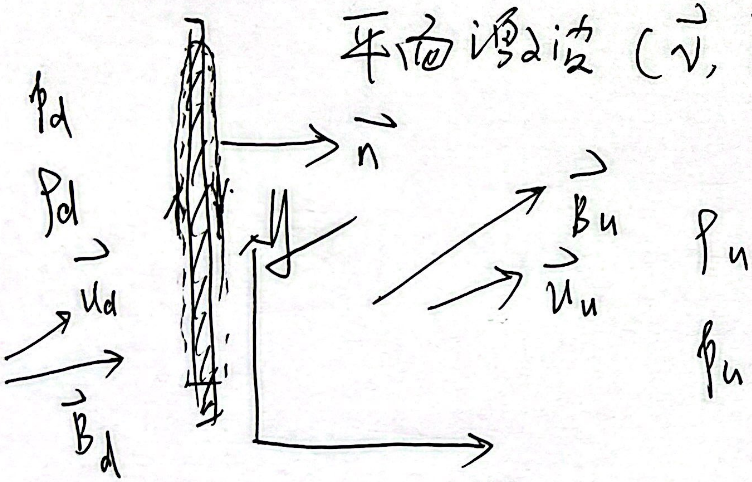
3.8.3 MHD shock.（磁场①），粒子、波相互作用②，耗散机制，波的形式③.

$$\begin{cases} \frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\vec{v}) = 0 \\ \frac{\partial(\rho\vec{v})}{\partial t} + \nabla\cdot\left[\rho\vec{v}\vec{v} + p\overleftrightarrow{I} + \frac{B^2}{2\mu_0}\overleftrightarrow{I} - \frac{\vec{B}\vec{B}}{\mu_0}\right] = 0 \end{cases}$$

$$\frac{\partial\left(\frac{p}{\gamma-1}+\frac{\rho v^2}{2}+\frac{B^2}{2\mu_0}\right)}{\partial t}+\nabla\cdot\left[\left(\frac{\gamma p}{\gamma-1}+\frac{\rho v^2}{2}\right)\vec{v}+\frac{\vec{E}\times\vec{B}}{\mu_0}\right]=0$$

$$\vec{E}+\vec{v}\times\vec{B}=0,\quad \frac{\partial\vec{B}}{\partial t}=\nabla\times(\vec{v}\times\vec{B}),\quad \nabla\cdot\vec{B}=0$$

平面激波 $(\vec{v},\vec{B},\vec{n})$.



$$\vec{v}=v_x\hat{e}_x+v_y\hat{e}_y=v_n\vec{n}+v_\tau\vec{\tau}$$

$$\vec{B}=B_x\hat{e}_x+B_y\hat{e}_y=B_n\hat{n}+B_\tau\hat{e}$$

$\Rightarrow$

① $\rho v_n\Big|_{u-d}=0$

② $\rho\vec{v}v_n+p+\frac{B^2}{2\mu_0}-\frac{\vec{B}B_n}{\mu_0}\Big|_{u-d}=0$

法向
切向

③ $\left[\left(\frac{\gamma p}{\gamma-1}+\frac{\rho v^2}{2}\right)v_n+\frac{\vec{E}\times\vec{B}}{\mu_0}\Big|_n\right]=0.$

$(\vec{v}\times\vec{B})\times\vec{B}$
$=-\vec{B}\times(\vec{v}\times\vec{B})$
$=-(\vec{B}\cdot\vec{B}\vec{v}-\vec{B}\cdot\vec{v}\vec{B})$
$=-(B^2\vec{v}-\vec{v}\cdot\vec{B}\vec{B})$

$\left[\cdots+B^2v_n-\vec{v}\cdot\vec{B}B_n\right]_{u-d}=0$

切向电场连续，$\vec{v}\times\vec{B}\Big|_y^{u-d}=0$ ④

法向磁场连续 . $B_n\Big|^{u-d}=0$. ⑤

$$\cancel{\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\vec{v}) = 0}$$

分别在上、下游写出等离子体满足的守恒关系：

$$\begin{cases} \cancel{\frac{\partial(\rho\vec{v})}{\partial t}} + \nabla\cdot\left[\rho\vec{v}\vec{v} + p\vec{I} + \frac{B^2}{2\mu_0}\vec{I} - \frac{\vec{B}\vec{B}}{\mu_0}\right] = 0 \\ \cancel{\frac{\partial}{\partial t}\left(\frac{p}{\gamma-1} + \frac{\rho v^2}{2} + \frac{B^2}{2\mu_0}\right)} + \nabla\cdot\left[\left(\frac{\gamma p}{\gamma-1} + \frac{\rho v^2}{2}\right)\vec{v} + \frac{\vec{E}\times\vec{B}}{\mu_0}\right] = 0 \\ \cancel{\vec{E} + \vec{v}\times\vec{B} = \frac{\partial\vec{B}}{\partial t}} \quad \vec{E} + \vec{v}\times\vec{B} = \eta\vec{j} = 0 \\ \cancel{\frac{\partial\vec{B}}{\partial t}} = \nabla\times(\vec{v}\times\vec{B}) = 0 \quad \nabla\cdot\vec{B} = 0 \end{cases}$$

$$\to -\frac{(\vec{v}\times\vec{B})\times\vec{B}}{\mu_0} = \frac{\vec{B}\cdot\vec{B}}{\mu_0}\vec{v} - \frac{\vec{B}\cdot\vec{v}\,\vec{B}}{\mu_0} = \frac{B^2}{\mu_0}\vec{v} - \frac{\vec{v}\cdot\vec{B}}{\mu_0}\vec{B}$$

$$\oint_S \nabla\times(\vec{v}\times\vec{B})\cdot d\vec{s} = \oint_L \vec{v}\times\vec{B}\cdot d\vec{l}$$

体积分 → 面积分 $(\cdot\vec{n})$

$\Rightarrow$ ① $[\rho v_n]_{u-d} = 0$ ， $\cancel{\rho_1 v_1 = \rho_2 v_2}$ ， $\rho_1 v_{x1} = \rho_2 v_{x2}$

② $\rho\vec{v}v_n + p\vec{n} + \frac{B^2}{2\mu_0}\vec{n} - \frac{\vec{B}B_n}{\mu_0}\Big|_{u-d} = 0$

$$\Rightarrow \rho(v_x\hat{e}_x + v_y\hat{e}_y)v_x + p\hat{e}_x + \frac{B^2}{2\mu_0}\hat{e}_x - \frac{(B_x\hat{e}_x + B_y\hat{e}_y)B_x}{\mu_0}\Big|^{1-2}_{u-d} = 0$$

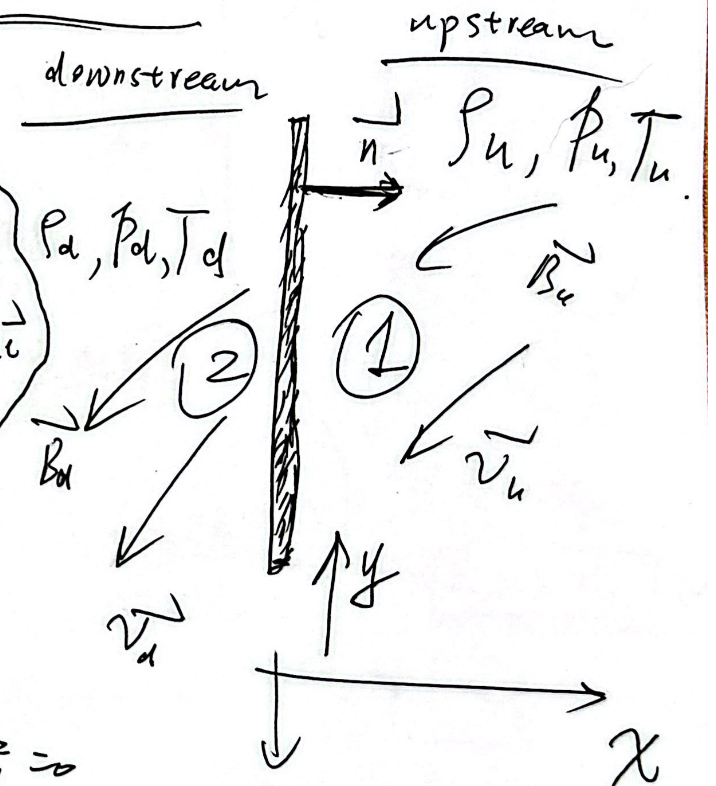
$$\begin{cases} \rho v_x^2 + p + \frac{B^2}{2\mu_0} - \frac{B_x^2}{\mu_0}\Big|_{u-d} = 0 \\ \rho v_x v_y - \frac{B_x B_y}{\mu_0}\Big|_{u-d} = 0 \end{cases}$$

③ $\left(\frac{\gamma p}{\gamma-1} + \frac{\rho v^2}{2}\right)v_x + \frac{B^2}{\mu_0}v_x - (v_x B_x + v_y B_y)/\mu_0\, B_x\Big|^{u-d} = 0$

$B_{x1} = B_{x2}$

$\vec{v}\times\vec{B}\Big|^{1-2}_{z} = \cancel{\vec{v}\times\vec{B}}\Big|_z = 0$

$\begin{pmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ v_x & v_y & 0 \\ B_x & B_y & 0 \end{pmatrix}$ $v_x B_y - v_y B_x\Big|_{u-d} = 0$



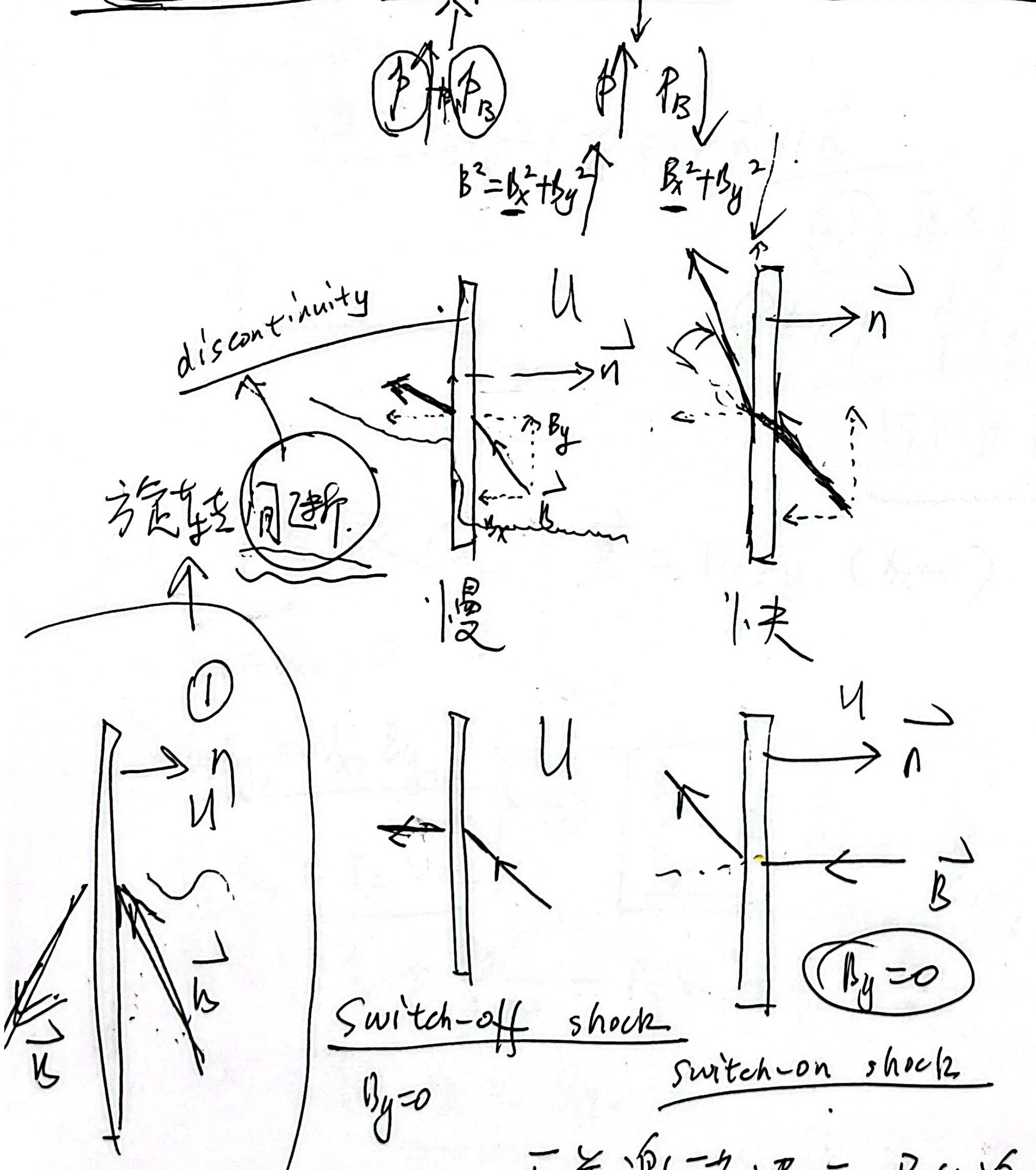
shock frame

$\vec{v} = v_x\hat{e}_x + v_y\hat{e}_y$

$\vec{B} = B_x\hat{e}_x + B_y\hat{e}_y$

① HD激波与MHD激波的区别：磁场，粒子加速，波波粒子作用 ⇒ 耗散机制 → 无碰撞激波

② 三组解：快激波、慢激波、中间激波（大振幅Alfvén波 —不可压）



开关激波：导出、导出激波

$$\Rightarrow \rho' = p' = 0$$

$$\rho\Big|^{u-d} = 0,\quad \rho_u = \rho_d$$

$$p_u = p_d,$$

$$v_{x1} = v_{x2}$$

$$B_{x1} = B_{x2}$$

$$\frac{\rho}{2}(v_y^2)v_x - \frac{(\cancel{v_x B_x} + v_y B_y)}{\mu_0}B_x$$

$$\frac{\rho_1 v_x v_{y1}^2}{2} - \frac{v_{y1} B_{x1} B_{y1}}{\mu_0} = \frac{\rho_2 v_{x2} v_{y2}^2}{2} - \frac{v_{y2} B_{x2} B_{y2}}{\mu_0}$$

$$\frac{\rho_1 v_{x1}(v_{y1}^2 - v_{y2}^2)}{2} = \frac{B_{x1}B_{y1}(v_{y1}+v_{y2})}{\mu_0}$$

$$\boxed{B_{y1}^2 = B_{y2}^2}$$

$$\Rightarrow B_{y1} = \pm B_{y2}$$

$$\Rightarrow \boxed{B_{y1} = -B_{y2}}$$

① $v_{y1} + v_{y2} = 0$

② $\frac{\rho_1 v_{x1}}{2}(v_{y1} - v_{y2}) = \frac{B_{x1}B_{y1}}{\mu_0}$

$$\Rightarrow v_{y1} + v_{y2} = v_{y1} + v_{y2}$$

$$v_{x1}B_{y1} - v_{y1}B_{x1}$$

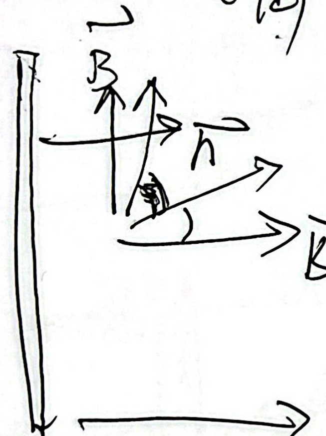
$$= v_{x2}B_{y2} - v_{y2}B_{x2}$$

$$= -v_{x1}B_{y1} - v_{y2}B_{x1}$$

$$\Rightarrow 2 v_{x1}B_{y1} = B_{x1}(v_{y1} - v_{y2})$$

$$\Rightarrow$$

③ 平行激波、垂直激波、倾斜激波.
(准) (准) Oblique



$\vec{B}$ (平行) $\vec{n} \parallel \vec{B}$

③.1 平行激波

$\rho(x)$. $p(x)$. $\vec{v}(x)$; $\vec{B}(x)$ $B_{x_1} = B_{x_2}$

与 HD 激波关系式一致.

$$\frac{\rho_2}{\rho_1} \le \frac{\gamma+1}{\gamma-1}.$$

③.2 垂直激波：$\vec{B} = B_y \hat{e}_y$ ($B_x = 0$).

$$\left\{\begin{array}{l} B_{x_1} = B_{x_2} = 0 \\ v_{x_1} B_{y_1} = v_{x_2} B_{y_2} \\ \rho_1 v_{x_1} = \rho_2 v_{x_2} \end{array}\right\} \Rightarrow \boxed{\frac{B_{y_2}}{B_{y_1}} = \frac{\rho_2}{\rho_1} = \frac{v_{x_1}}{v_{x_2}}}$$

$$\rho_1 v_{x_1}^2 + p_1 + \frac{B_{y_1}^2}{2\mu_0} = \rho_2 v_{x_2}^2 + p_2 + \frac{B_{y_2}^2}{2\mu_0}$$

$$v_{y_1} = v_{y_2}$$

$$\left(\frac{\gamma p_1}{\gamma-1} + \frac{\rho_1 v_1^2}{2}\right) v_{x_1} + \frac{B_{y_1}^2}{\mu_0} v_{x_1} = \left(\frac{\gamma p_2}{\gamma-1} + \frac{\rho_2 v_2^2}{2}\right) v_{x_2} + \frac{B_{y}^2}{\mu_0} v_{x_2}$$

$$\Rightarrow \boxed{2(2-\gamma)X^2 + [2\beta_1 + 2 + (\gamma-1)M_1^2\beta_1]\gamma X - \gamma(\gamma+1)M_1^2\beta_1 = 0}.$$

$$\Rightarrow \boxed{X \le \frac{\gamma+1}{\gamma-1}}$$

推导

$$M_1 = \frac{v_1}{c_{s1}},\quad \beta_1 = \frac{p_1}{\frac{B_{y_1}^2}{2\mu_0}},\quad c_{s1}^2 = \frac{\gamma p_1}{\rho_1}$$

$$\frac{v_{A_1}^2}{2c_{s1}^2} = \frac{1}{\gamma\beta_1}\qquad \frac{p_1}{\rho_1 v_1^2} = \frac{1}{\gamma M_1^2}$$

④ 激波粒子加速 (电场)：

① 平行激波 ② 垂直激波.

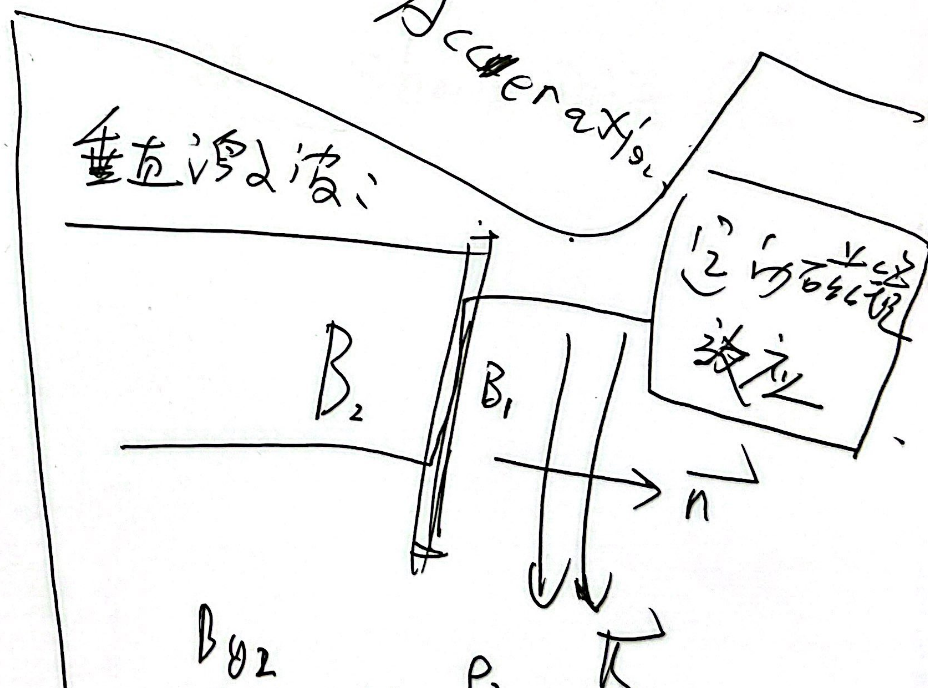
上下游存在强湍流；=> 二阶 Fermi 加速.

$$\Delta v = v_u - v_d$$

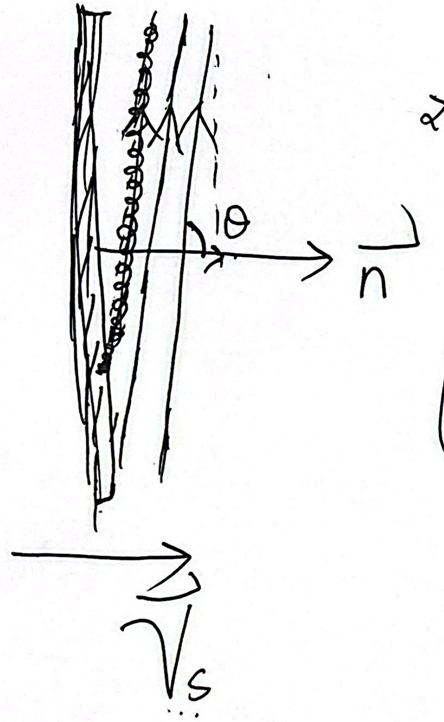
DSA：

Diffusive Shock Acceleration

垂直激波：



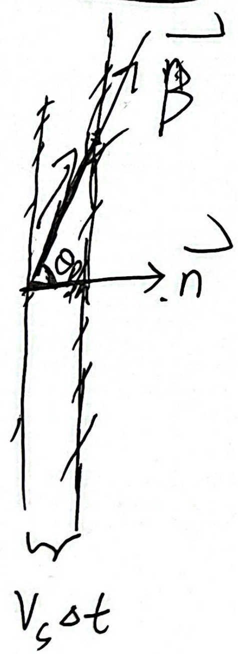
$$\frac{B_{y2}}{B_{y1}} = X = \frac{\rho_2}{\rho_1}.$$



Fermi加速：$\Delta E = 2mv^2\left[\frac{V_m}{v} + \left(\frac{V_m}{v}\right)^2\right]$

对沿$\vec{B}$运动的电子而言，shock速度？

$\boxed{\frac{B_{min}}{B_{max}}}$



$\vec{V}_s$

$= \left|\frac{V_s\Delta t}{\cos\theta_{Bn}}\right| >> |V_s\Delta t|$

→ 快Fermi加速

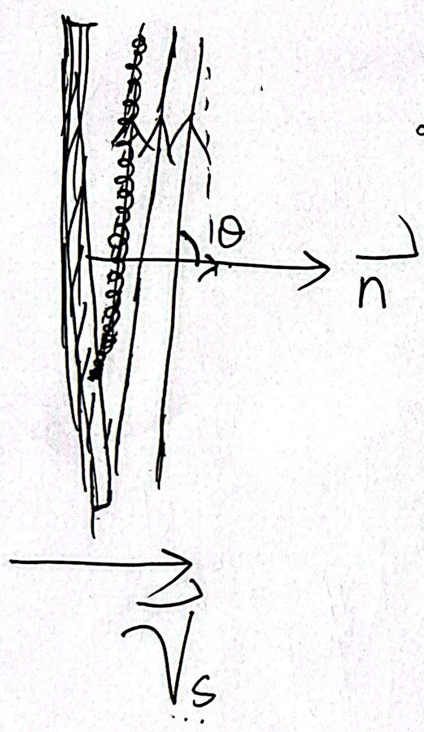
吴京生

激波漂移加速

SDA: Shock Drift Acc.

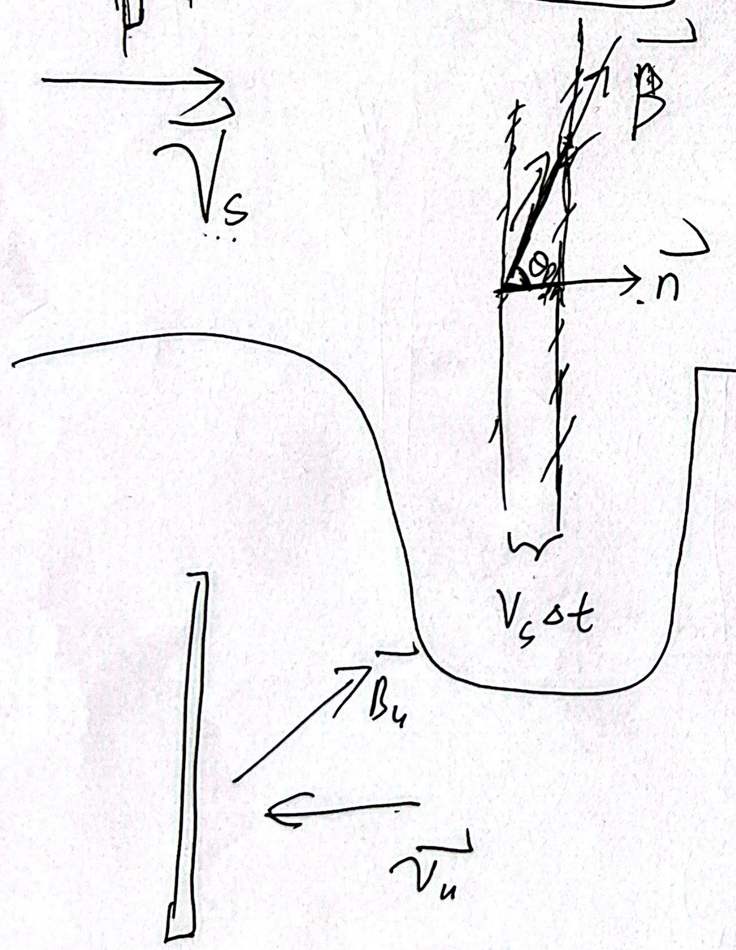
Fermi加速: $\Delta E = 2mv^2\left[\frac{V_m}{v} + \left(\frac{V_m}{v}\right)^2\right]$

对沿$\vec{B}$之运动电子而言，shock速度?



$\boxed{\frac{B_{min}}{B_{max}}}$

V_s



$\left|\frac{V_s \Delta t}{\cos\theta_{Bn}}\right| >> |V_s \Delta t|$

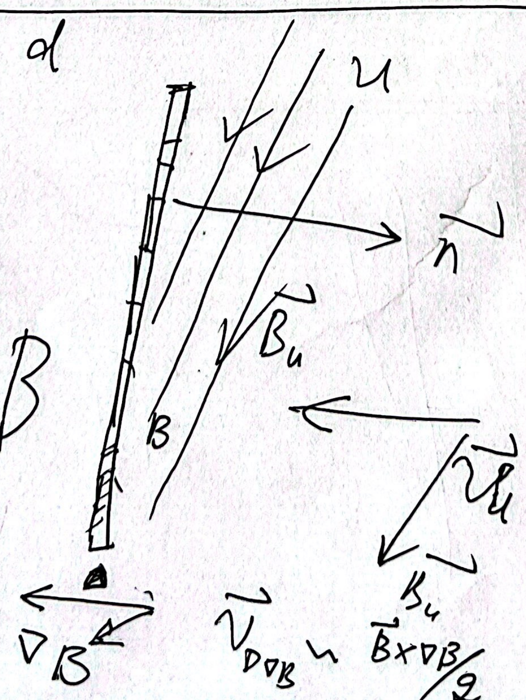
快Fermi加速

吴京生.

激波漂移加速

SDA: Shock Drift Acc.

$\boxed{\text{主要是电子加速}}$



对流电场

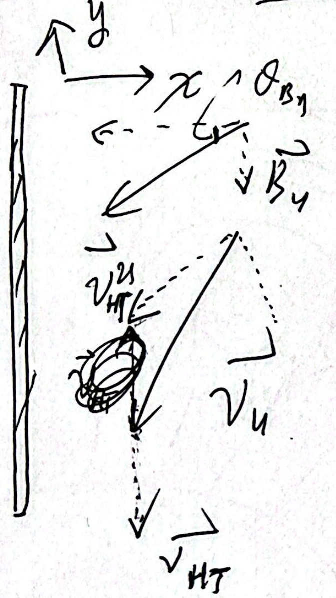
$\vec{E}_u = -\vec{v}_u \times \vec{B}_u$: 向内

∇B方向.

电子: 向外. $q\vec{E}_u$ 向外

⑤ De Hoffmann-Teller (HT) 参考系．（请去上下游对流电场，简化推导）

$$\vec{E}^{HT} = -\vec{v}_u^{HT} \times \vec{B}_u = 0 \quad (\vec{v}_u^{HT} \parallel \vec{B}_u) \qquad \vec{v}^{HT}, \quad \boxed{\vec{v}_u - \vec{v}^{HT} = \vec{v}_u^{HT}}$$



$$(\vec{v}_u - \vec{v}^{HT}) \times \vec{B}_u = 0$$

$$\Rightarrow \vec{v}_u \times \vec{B}_u = \vec{v}^{HT} \times \vec{B}_u$$

$$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{A}\cdot\vec{C}\,\vec{B} - \vec{A}\cdot\vec{B}\,\vec{C}$$

$$(\vec{v}_u \times \vec{B}_u) \times \vec{n} = \vec{n}\cdot\vec{v}_u\,\vec{B}_u - \vec{n}\cdot\vec{B}_u\,\vec{v}_u$$

$$\boxed{\vec{v}_{HT} = \frac{\vec{n} \times (\vec{v}_u \times \vec{B}_u)}{\vec{n}\cdot\vec{B}_u}} = \frac{\vec{n}\cdot\vec{B}_u\,\vec{v}_u - \vec{n}\cdot\vec{v}_u\,\vec{B}_u}{\vec{n}\cdot\vec{B}_u}$$

$$\Rightarrow \vec{v}_{HT} \times \vec{B}_u = \vec{v}_u \times \vec{B}_u$$

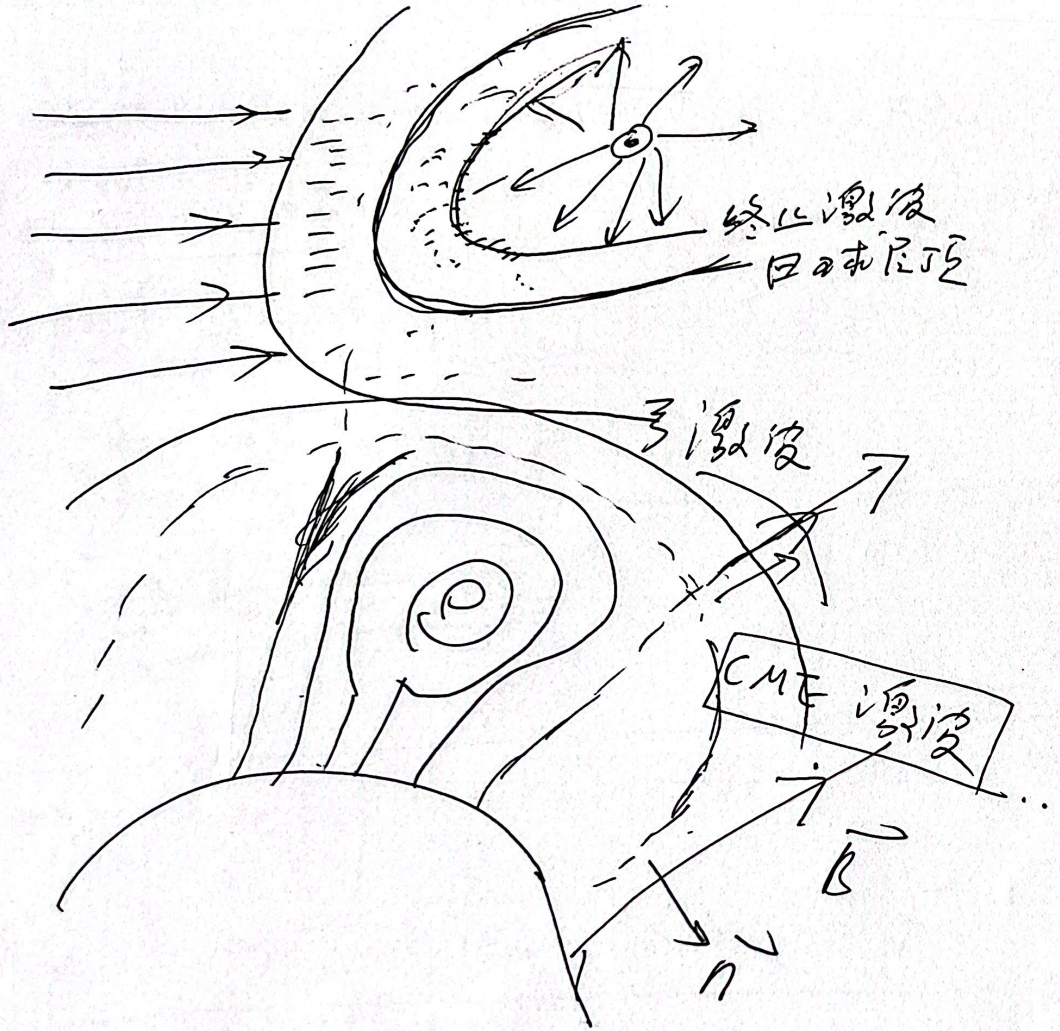
$$\vec{v}_{HT} = \frac{B_{ux}\vec{v}_u - v_{ux}\vec{B}_u}{B_{ux}} = \frac{\cancel{B_{ux}v_{ux}\hat{e}_x} - B_{ux}v_{uy}\hat{e}_y - \cancel{v_{ux}B_{ux}\hat{e}_x} - v_{ux}B_{uy}\hat{e}_y}{B_{ux}} = \left(v_{uy} - v_{ux}\frac{B_{uy}}{B_{ux}}\right)$$

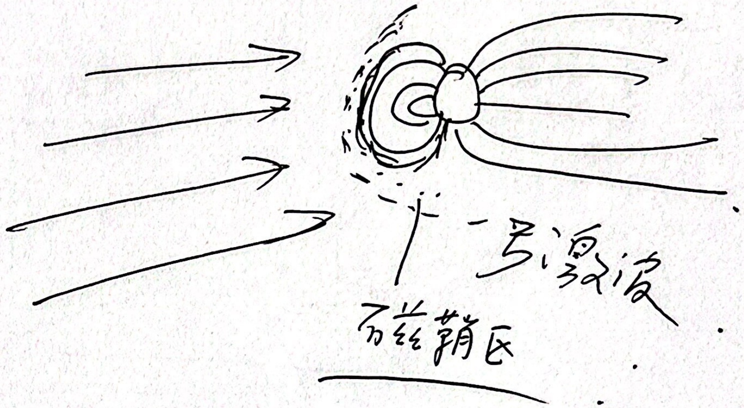
E. R. Priest

Solar MHD

MHD of the Sun

$$v_{HT} = v_{uy} - v_{ux}\frac{B_{uy}}{B_{ux}} = v_{uy} - v_{ux}\,\mathrm{tg}\,\theta_{Bn}$$





Crab Nebula

客星

1054年

$$\begin{cases} \frac{\partial n_e}{\partial t} + \nabla\cdot(n_e\vec{v}_e) = 0 . \quad \frac{\partial n_i}{\partial t} + \nabla\cdot(n_i\vec{v}_i) = 0 . \\ \rho_e\left(\frac{\partial \vec{v}_e}{\partial t} + \vec{v}\cdot\nabla\vec{v}_e\right) = \rho_e\frac{d\vec{v}_e}{dt} = -\nabla p_e - e n_e(\vec{E} + \vec{v}_e\times\vec{B}) \\ \rho_i \frac{d\vec{v}_i}{dt} = -\nabla p_i + e n_i(\vec{E} + \vec{v}_i\times\vec{B}) \\ p_e = C_e\rho_e^{\gamma_e} , \quad p_i = C_i\rho_i^{\gamma_i} . \end{cases}$$

$$\vec{j} = \sum_\alpha n_\alpha q_\alpha \vec{v}_\alpha .$$

$$\rho_q = \sum_\alpha n_\alpha q_\alpha .$$

$$\cancel{\frac{\partial \vec{B}}{\partial t} = \nabla\times\vec{E} , \frac{\partial\vec{E}}{\partial t} =}$$

$$\boxed{\begin{gathered} \nabla\cdot\vec{E} = \rho_q/\varepsilon_0 \quad \nabla\cdot\vec{B} = 0 \\ \nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} , \nabla\times\vec{B} = \mu_0\vec{j} + \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t} \end{gathered}}$$

$-\nabla p_e$	$-en_e\vec{E}$	$-e\vec{v}_e\times\vec{B}$	$-\nabla p_i$	$en_i\vec{E}$	$en_i\vec{v}_i\times\vec{B}$	$\partial\vec{B}/\partial t$	$\partial\vec{E}/\partial t$
$k^2 c_{se}^2$	ω_{pe}^2	ω_{ce}	$k^2 c_{si}^2$	ω_{pi}^2	ω_{pi} ω_{ci}		k^2c^2

(全等离子体)

~~电子~~ Langmuir 波 → 等离子体振荡 $\omega^2 = \omega_{pe}^2$, $\sum_\alpha n_{\alpha 0} q_\alpha = 0$

$$\left[\begin{aligned} &\frac{\partial n_\alpha'}{\partial t} + n_{\alpha 0}\nabla\cdot(\cancel{n_\alpha}\vec{V}_\alpha) = 0 , \quad \rho_{\alpha 0}\frac{\partial\vec{v}_\alpha}{\partial t} = q_\alpha n_{\alpha 0}\vec{E} \\ &\nabla\cdot\vec{E} = \sum_\alpha n_\alpha q_\alpha/\varepsilon_0 \qquad \frac{\partial}{\partial t}(\nabla\cdot\vec{v}_\alpha) = \frac{q_\alpha}{m_\alpha}\nabla\cdot\vec{E} = -\frac{\partial^2}{\partial t^2} n_\alpha'/n_{\alpha 0} \end{aligned}\right.$$

$$i\vec{k}\cdot\vec{E} = \sum_\alpha \frac{n_\alpha' q_\alpha}{\varepsilon_0}, \quad \frac{n_{\alpha 0} q_\alpha}{\omega^2 m_\alpha} i\vec{k}\cdot\vec{E} = n_\alpha' \Rightarrow i\vec{k}\cdot\vec{E} = \sum_\alpha \frac{n_\alpha q_\alpha^2}{\varepsilon_0 m_\alpha} \frac{i\vec{k}\cdot\vec{E}}{\omega^2}$$

$$\Rightarrow \boxed{\omega^2 = \sum_\alpha \omega_{p\alpha}^2} \boxed{\omega^2 \approx \omega_{pe}^2}. \quad \omega_{pi}^2 \ll \omega_{pe}^2.$$

4.2.1 (电介质) 色散关系的一般形式 (冷等离子体)

$$\frac{\partial \vec{B}}{\partial t} = -\nabla\times\vec{E}, \quad \nabla\times\vec{B} = \mu_0\vec{j} + \mu_0\varepsilon_0\frac{\partial \vec{E}}{\partial t}, \quad \boxed{\vec{j} = \overleftrightarrow{\sigma}\cdot\vec{E} = \sum_\alpha n_\alpha q_\alpha \vec{v}_\alpha}$$

电导率张量 (1阶量级)

$$-i\omega\vec{B} = -i\vec{k}\times\vec{E}, \quad i\vec{k}\times\vec{B} = \mu_0\vec{j} - i\mu_0\varepsilon_0\omega\vec{E} = (\mu_0\overleftrightarrow{\sigma} - i\mu_0\varepsilon_0\omega\overleftrightarrow{I})\cdot\vec{E}$$

$$\boxed{\overleftrightarrow{\varepsilon} = \overleftrightarrow{I} + \frac{\mu_0\overleftrightarrow{\sigma}}{-i\mu_0\varepsilon_0\omega} = \overleftrightarrow{I} + \frac{i\overleftrightarrow{\sigma}}{\omega\varepsilon_0}}$$

$-i\mu_0\varepsilon_0\omega\overleftrightarrow{\varepsilon}\cdot\vec{E}$

介电张量

$$\begin{cases} \omega\vec{B} = \vec{k}\times\vec{E} \\ \vec{k}\times\vec{B} = -\mu_0\varepsilon_0\omega\overleftrightarrow{\varepsilon}\cdot\vec{E} \end{cases} \Rightarrow \vec{k}\times(\vec{k}\times\vec{E}) = \omega\vec{k}\times\vec{B} = -\mu_0\varepsilon_0\omega^2\overleftrightarrow{\varepsilon}\cdot\vec{E} = -\frac{\omega^2}{c^2}\overleftrightarrow{\varepsilon}\cdot\vec{E}$$

$$\Rightarrow (\vec{k}\cdot\vec{E})\vec{k} - k^2\vec{E} + \frac{\omega^2}{c^2}\overleftrightarrow{\varepsilon}\cdot\vec{E} = 0$$

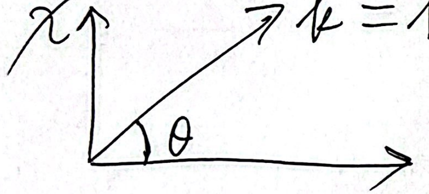
$$\left(\vec{k}\vec{k} - k^2\overleftrightarrow{I} + \frac{\omega^2}{c^2}\overleftrightarrow{\varepsilon}\right)\cdot\vec{E} = 0$$

$$\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}\begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = 0$$

关于 E_x, E_y, E_z 的三个一次代数方程组.

$$\vec{k}\vec{k} = \begin{pmatrix} k_x \\ 0 \\ k_z \end{pmatrix} (k_x, 0, k_z) = \begin{pmatrix} k_x^2, & 0, & k_x k_z \\ 0, & 0, & 0 \\ k_z k_x, & 0, & k_z^2 \end{pmatrix}. \qquad \begin{pmatrix} k^2 & 0 & 0 \\ 0 & k^2 & 0 \\ 0 & 0 & k^2 \end{pmatrix}, \quad \overleftrightarrow{\varepsilon} \quad \overleftrightarrow{\sigma}$$

$$\vec{k} = k_x \hat{e}_x + k_z \hat{e}_z = k \sin\theta \, \hat{e}_x + k \cos\theta \, \hat{e}_z$$



$\vec{B}_0(z)$

有非零解的条件： $\det \left| \vec{k}\vec{k} - k^2 \overleftrightarrow{I} + \frac{\omega^2}{c^2} \overleftrightarrow{\varepsilon} \right| = 0$

$$\boxed{\sum_\alpha n_\alpha q_\alpha \vec{v}_\alpha = \overleftrightarrow{\sigma} \cdot \vec{E}}.$$

$$\begin{pmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ v_{\alpha x} & v_{\alpha y} & v_{\alpha z} \\ 0 & 0 & B_0 \end{pmatrix} =$$

FT

$$n_{\alpha 0} m_\alpha \frac{d\vec{v}_\alpha}{dt} = + q_\alpha n_\alpha (\vec{E} + \vec{v}_\alpha \times \vec{B}_0) \quad n_{\alpha 0} m_\alpha (-i\omega \vec{v}_\alpha) = q_\alpha n_\alpha (\vec{E} + \vec{v}_\alpha \times \vec{B}_0).$$

$$\overleftrightarrow{A}_\alpha \cdot \vec{v}_\alpha = \vec{E}$$

$$\begin{cases} -i\omega n_{\alpha 0} m_\alpha v_{\alpha x} = q_\alpha n_\alpha E_x + q_\alpha n_\alpha v_{\alpha y} B_0 \\ -i\omega n_{\alpha 0} m_\alpha v_{\alpha y} = q_\alpha n_\alpha E_y - q_\alpha n_\alpha v_{\alpha x} B_0 \\ -i\omega n_{\alpha 0} m_\alpha v_{\alpha z} = q_\alpha n_\alpha E_z. \end{cases}$$

$$\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix} \begin{pmatrix} v_{\alpha x} \\ v_{\alpha y} \\ v_{\alpha z} \end{pmatrix} = \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}.$$

$$\vec{v}_\alpha = \overleftrightarrow{A}_\alpha^{-1}\cdot\vec{E} \Rightarrow \sum_\alpha n_\alpha q_\alpha \overleftrightarrow{A}_\alpha^{-1}\cdot\vec{E} = \overleftrightarrow{\sigma}\cdot\vec{E}$$

$$\Rightarrow \overleftrightarrow{\sigma} = \sum_\alpha n_\alpha q_\alpha \overleftrightarrow{A}_\alpha^{-1}$$

$$\boxed{\overleftrightarrow{\varepsilon} = \overleftrightarrow{I} + \frac{i\overleftrightarrow{\sigma}}{\omega\varepsilon_0}}$$

$$\det\left|\vec{k}\vec{k} - k^2\overleftrightarrow{I} + \frac{\omega^2}{c^2}\overleftrightarrow{\varepsilon}\right| = 0$$

$$n = \frac{kc}{\omega}, \quad \vec{n} = \frac{\vec{k}c}{\omega}$$

$$= c/v_p.$$

折射率矢量

$$\det\left|\vec{n}\vec{n} - n^2\overleftrightarrow{I} + \overleftrightarrow{\varepsilon}\right| = 0$$

$$\vec{n} = n\sin\theta\,\hat{e}_x + n\cos\theta\,\hat{e}_z$$

$$\Rightarrow An^4 - Bn^2 + C = 0.$$

$$n^2 = \frac{B \pm \sqrt{B^2-4AC}}{2A} = \frac{k^2c^2}{\omega^2}.$$

推导§4.2节

$$S = 1 - \sum_\alpha \frac{\omega_{p\alpha}^2}{\omega^2 - \omega_{c\alpha}^2}$$

$$D = \sum_\alpha \frac{\omega_{p\alpha}^2\,\omega_{c\alpha}}{\omega(\omega^2 - \omega_{c\alpha}^2)}.$$

$$C = PRL.$$

$$P = 1 - \sum_\alpha \frac{\omega_{p\alpha}^2}{\omega^2}, \quad R = S+D, \quad L = S-D.$$

第四章 双流体中的等离子体波动现象

$$\begin{cases} \dfrac{\partial n_e}{\partial t} + \nabla\cdot(n_e \vec{v}_e) = 0, \quad \dfrac{\partial n_i}{\partial t} + \nabla\cdot(n_i \vec{v}_i) = 0, \\ \rho_e \dfrac{d\vec{v}_e}{dt} = -\nabla p_e + (-e) n_e (\vec{E} + \vec{v}_e \times \vec{B}) \\ \rho_i \dfrac{d\vec{v}_i}{dt} = -\nabla p_i + e n_i (\vec{E} + \vec{v}_i \times \vec{B}) \\ p_e = C_e \rho_e^{\gamma_e}, \quad p_i = C_i \rho_i^{\gamma_i} \end{cases}$$

$$\nabla\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla\times\vec{B} = \mu_0 \vec{j} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} = \mu_0 \overleftrightarrow{\sigma}\cdot\vec{E} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

$$\nabla\cdot\vec{E} = \rho_q/\varepsilon_0$$

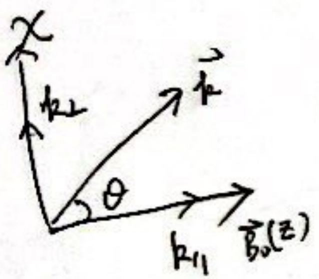
$$\nabla\cdot\vec{B} = 0$$

$$\vec{j} = \sum_\alpha n_\alpha q_\alpha \vec{v}_\alpha, \quad \rho_q = \sum_\alpha n_\alpha q_\alpha$$

4.2 一般形式: $\underline{\vec{j} = \overleftrightarrow{\sigma}\cdot\vec{E}}$, $\overleftrightarrow{\sigma}$ 电导率张量

$$\vec{k}\times\vec{E} = \omega\vec{B}, \quad \vec{k}\times\vec{B} = \frac{\vec{k}}{\omega}\times(\vec{k}\times\vec{E}) = -\omega\mu_0\varepsilon_0 \overleftrightarrow{\varepsilon}\cdot\vec{E}, \quad \overleftrightarrow{\varepsilon} = \overleftrightarrow{I} + \frac{i\overleftrightarrow{\sigma}}{\omega\varepsilon_0}$$

$$\Rightarrow \left(\frac{\omega^2}{c^2}\overleftrightarrow{\varepsilon} + \vec{k}\vec{k} - k^2\overleftrightarrow{I}\right)\cdot\vec{E} = 0 \quad \text{------ 电场}\ \vec{E}\ \text{的三分量之方程}$$



非零解条件 $\det|\overleftrightarrow{\Lambda}| = \det\left|\frac{\omega^2}{c^2}\overleftrightarrow{\varepsilon} + \vec{k}\vec{k} - k^2\overleftrightarrow{I}\right| = 0$,

令 $\vec{n} = \frac{\vec{k}c}{\omega}$, 得 $\det\left|\overleftrightarrow{\varepsilon} + \vec{n}\vec{n} - n^2\overleftrightarrow{I}\right| = 0$. $\quad n = \frac{kc}{\omega} = \frac{c}{v_p}$.

$\vec{k} = (k\cos\theta, k\sin\theta)$

$\vec{n} = (n\cos\theta, n\sin\theta)$

$$\vec{n}\vec{n} - n^2\overleftrightarrow{I} = \begin{pmatrix} -n^2\sin^2\theta & 0 & n^2\sin\theta\cos\theta \\ 0 & -n^2 & 0 \\ n^2\sin\theta\cos\theta & 0 & -n^2\cos^2\theta \end{pmatrix}$$

$$\overleftrightarrow{\varepsilon} = \overleftrightarrow{I} + \frac{i\overleftrightarrow{\sigma}}{\omega\varepsilon_0}, \quad \vec{j} = \sum_\alpha n_\alpha q_\alpha \vec{v}_\alpha = \overleftrightarrow{\sigma}\cdot\vec{E}$$

④

冷等离子体 $\rho_\alpha \frac{d\vec{v}_\alpha}{dt} = q_\alpha n_\alpha(\vec{E} + \vec{v}_\alpha \times \vec{B}) \sim -i n_{\alpha 0} m_\alpha \omega \vec{v}_\alpha = q_\alpha n_{\alpha 0}(\vec{E} + \vec{v}_\alpha \times \vec{B}_0) \Rightarrow \overleftrightarrow{A}_\alpha \cdot \vec{v}_\alpha = \frac{i q_\alpha}{m_\alpha \omega} \vec{E}$

$\Rightarrow \overleftrightarrow{A}_\alpha^{-1} \cdot \overleftrightarrow{A}_\alpha \cdot \vec{v}_\alpha = \frac{i q_\alpha}{m_\alpha \omega} \overleftrightarrow{A}_\alpha^{-1} \cdot \vec{E} \Rightarrow \vec{v}_\alpha = \frac{i q_\alpha}{m_\alpha \omega} \overleftrightarrow{A}_\alpha^{-1} \cdot \vec{E}$, $\quad \overleftrightarrow{A}_\alpha^{-1} = \frac{1}{\left(1 - \frac{\omega_{c\alpha}^2}{\omega^2}\right)} \begin{pmatrix} 1 & \frac{i\omega_{c\alpha}}{\omega} & 0 \\ -i\frac{\omega_{c\alpha}}{\omega} & 1 & 0 \\ 0 & 0 & 1 - \frac{\omega_{c\alpha}^2}{\omega^2} \end{pmatrix}$

$\overleftrightarrow{\sigma} = \sum_\alpha \overleftrightarrow{\sigma}_\alpha$, $\quad \overleftrightarrow{\sigma}_\alpha = \frac{i n_{\alpha 0} q_\alpha^2}{m_\alpha \omega} \overleftrightarrow{A}_\alpha^{-1}$, $\quad \overleftrightarrow{\varepsilon} = \overleftrightarrow{I} + \frac{i \overleftrightarrow{\sigma}}{\omega \varepsilon_0}$

$$\frac{\overleftrightarrow{\sigma}_\alpha}{\omega \varepsilon_0} = \frac{i \omega_{p\alpha}^2}{\omega^2 - \omega_{c\alpha}^2} \begin{pmatrix} 1 & \frac{i\omega_{c\alpha}}{\omega} & 0 \\ -i\frac{\omega_{c\alpha}}{\omega} & 1 & 0 \\ 0 & 0 & 1 - \frac{\omega_{c\alpha}^2}{\omega^2} \end{pmatrix}$$

$$\overleftrightarrow{\varepsilon} = \begin{pmatrix} S & -iD & 0 \\ iD & S & 0 \\ 0 & 0 & P \end{pmatrix}$$

$$S = 1 - \sum_\alpha \frac{\omega_{p\alpha}^2}{\omega^2 - \omega_{c\alpha}^2}$$

$$D = \sum_\alpha \frac{\omega_{p\alpha}^2 \omega_{c\alpha}}{\omega(\omega^2 - \omega_{c\alpha}^2)}$$

$$P = 1 - \sum_\alpha \frac{\omega_{p\alpha}^2}{\omega^2}$$

由 $\det\left|\vec{n}\vec{n} - n^2 \overleftrightarrow{I} + \overleftrightarrow{\varepsilon}\right| = 0$，得

$$\det \begin{vmatrix} S - n^2\cos^2\theta & -iD & n^2\sin\theta\cos\theta \\ iD & S - n^2 & 0 \\ n^2\sin\theta\cos\theta & 0 & P - n^2\sin^2\theta \end{vmatrix} = 0 \Rightarrow An^4 - Bn^2 + C = 0$$

$$\begin{cases} A = P\cos^2\theta + S\sin^2\theta \\ B = SP(1+\cos^2\theta) + (S^2 - D^2)\sin^2\theta \\ C = P(S^2 - D^2) = P(S+D)(S-D) = PRL \end{cases}$$

$$\boxed{\begin{matrix} R = S + D \\ L = S - D \end{matrix}}$$

可予全记 $B^2 - 4AC \geq 0$， $\Rightarrow \tan^2\theta = -\frac{P(n^2 - R)(n^2 - L)}{(n^2 - P)(Sn^2 - RL)}$ (Alice形式)

1° 截止(cut-off)频率与共振(resonance)频率 $(n^2=0,\ n^2\to\infty)$. ③

由 $n^2=0$，得 $C=0 \Rightarrow P=0,\ R=0,\ L=0$. 分别解得 $\omega=\omega_{pe},\ \omega=\omega_R,\ \omega=\omega_L$.

(电磁横波—光波).

对应于 O 模截止频率、右旋截止频率、左旋 …,

忽略离子运动：$\begin{cases} \omega_R = \dfrac{\Omega_{ce}}{2}+\sqrt{\omega_{pe}^2+\Omega_{ce}^2/4} \\ \omega_L = -\dfrac{\Omega_{ce}}{2}+\sqrt{\omega_{pe}^2+\Omega_{ce}^2/4} \end{cases}$.

$\cancel{R=0 \Rightarrow 1-\dfrac{\omega_{pe}^2}{\omega^2-\omega_{ce}^2}-\dfrac{\omega_{pe}^2\omega_{ce}}{\omega(\omega^2-\omega_{ce}^2)}=0}$

$\cancel{\omega(\omega^2-\omega_{ce}^2)-\omega_{pe}^2\omega-\omega_{ce}\omega_{pe}^2=0}$

$\cancel{\omega(\omega+\omega_{ce})-\omega_{pe}^2=0}$

$\cancel{\omega^2-\omega\omega_{ce}-\omega_{pe}^2=0}$

$\cancel{\omega_L=\dfrac{\omega_{ce}\pm\sqrt{\omega_{ce}^2+4\omega_{pe}^2}}{2}}$

共振频率 $n^2\to\infty \Rightarrow A=0 \Rightarrow P\cos^2\theta+S\sin^2\theta=0$

忽略离子运动：$\left(1-\dfrac{\omega_{pe}^2}{\omega^2}\right)\cos^2\theta+\left(1-\dfrac{\omega_{pe}^2}{\omega^2-\omega_{ce}^2}\right)\sin^2\theta=0$

$\theta=0°$时. $\omega\sim\omega_{pe}$.

$\theta=90°$时. $\omega\sim\omega_{pe}^2+\omega_{ce}^2=\omega_{UH}^2$，高杂共振. } 一般情况 $\omega_\pm(\omega_\Sigma,\omega_W)$ (自行求解绘图)

$R=S+D=1-\sum_\alpha\dfrac{\omega_{p\alpha}^2}{\omega^2-\omega_{c\alpha}^2}+\sum_\alpha\dfrac{\omega_{p\alpha}^2\omega_{c\alpha}}{\omega(\omega^2-\omega_{c\alpha}^2)}$

$=1-\sum_\alpha\dfrac{\omega\omega_{p\alpha}^2-\omega_{p\alpha}^2\omega_{c\alpha}}{\omega(\omega^2-\omega_{c\alpha}^2)}=1-\sum_\alpha\dfrac{\omega_{p\alpha}^2}{\omega(\omega+\omega_{c\alpha})}$

$R=0 \Rightarrow 1-\dfrac{\omega_{pe}^2}{\omega(\omega-\Omega_{ce})}=0$

$\omega^2-\omega\Omega_{ce}-\omega_{pe}^2=0$

$\omega_R=\dfrac{\Omega_{ce}\pm\sqrt{\Omega_{ce}^2+4\omega_{pe}^2}}{2}$ (去掉负值)

$L=S-D=\cdots$

2° 平行传播 $\theta=0°$. $P=0,\ n^2=R,\ n^2=L$.

(三种模式) $\omega^2=\omega_{pe}^2$, $\begin{cases}\dfrac{k^2c^2}{\omega^2}=1-\dfrac{\omega_{pe}^2}{(\omega+\omega_{ce})\omega} \\ \dfrac{k^2c^2}{\omega^2}=1-\dfrac{\omega_{pe}^2}{(\omega-\omega_{ce})\omega}\end{cases}$

垂直传播 $\theta=90°$.

$n^2=P,\quad n^2=RL/S$.

O模 $\dfrac{k^2c^2}{\omega^2}=1-\dfrac{\omega_{pe}^2}{\omega^2} \Rightarrow \omega^2=\omega_{pe}^2+k^2c^2$

X模 $\dfrac{k^2c^2}{\omega^2}=RL/S$，一般情况：$n^2=\dfrac{1}{2A}(B\pm\sqrt{B^2-4AC})$

冷等离子体磁离子理论 (Magnetoionic theory of waves in cold plasmas) : X, O, Z, W ②

1° 冷等离子体假设. 2° 离子静止不动 —— 考虑 $\omega \gg \omega_{pi}, \Omega_i$ (高频模式) ⇒ Appleton-Hartree (A-H) 形式

$n^2 = \frac{1}{2A}(B \pm \sqrt{B^2-4AC})$. $An^2 + An^4 - Bn^2 + C = An^2 \Rightarrow n^2 = \frac{An^2-C}{An^2-B+A}$

代入得: $N^2 = 1 - \frac{X}{Q}$, $X = \frac{\omega_{pe}^2}{\omega^2}$, $Y = \frac{\Omega_e}{\omega}$, $Q = 1 - \frac{Y^2\sin^2\theta}{2(1-X)} \pm \frac{X-1}{|X-1|}\left[\frac{Y^4\sin^4\theta}{4(1-X)^2} + Y^2\cos^2\theta\right]^{1/2}$

则可解出 n^2 $\left(\frac{k^2c^2}{\omega^2} = f(\omega)\right)$, 有两组解, 对应于 "+", "−". $\left\{\begin{array}{l} \text{"+"}: X, Z \\ \text{"−"}: O, W \end{array}\right\}$ 一快一慢 (一高频, 一低频) 相速度 (同一 (k, θ))

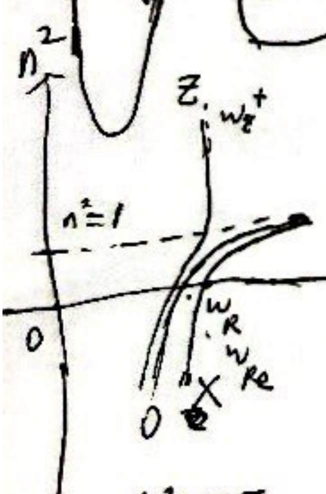
X. O 模 —— 右. 左旋快电磁波 (光波).

—— 可逃逸模式. (有截止频率: ω_{pe}, ω_R) (escaping modes)

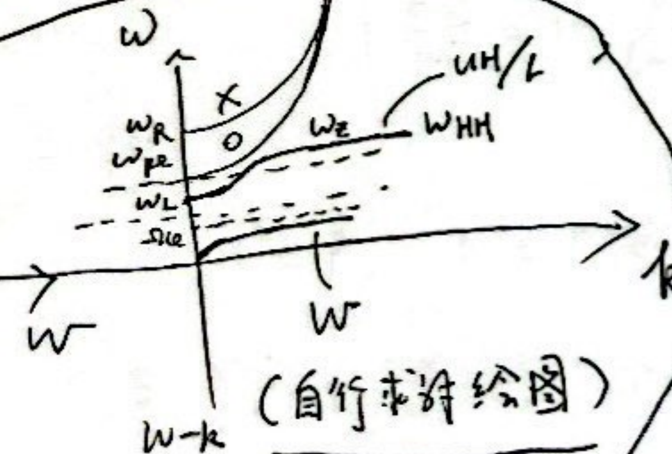
Z. W 模. —— Z 模与 Langmuir 波. 高杂波 (UH/HH) 位于同支色散关系.

—— 束缚模式. (trapped modes), 有截止、有共振: $\left\{\begin{array}{l} Z: \omega_L, \omega_Z, (\omega_+) \\ W: 0, \omega_W (\omega_-) \end{array}\right.$

ω_W^- 哨声模



$n^2-\omega$



$\omega-k$ (自行求解绘图)

比较n个特征频率间的大小关系:

$\boxed{\text{一般情况下的磁离子波色散关系曲线 } X\ O\ Z\ W}$ ① $\omega-k$ 曲线 ② $n^2-\omega$ 曲线

求得色散关系后, 可根据 $\vec{k}\cdot\vec{E}=0$ 解出各模式的场分量间大小及相位关系!

$\left\{\begin{array}{l} n^2>0: \text{传播区 (高频, 低频)} \\ n^2=0: \text{截止处} \\ n^2<0: \text{禁止区 (不传播区)} \end{array}\right.$

⑥

4.3 非磁化等离子体波动模式

($\Omega_e=0$). $\vec{B}_0=0$，可取 $\theta=0$，($\vec{k}=k\hat{e}_z$)，得：

$S=P=R=L=1-\sum_\alpha \frac{\omega_{p\alpha}^2}{\omega^2}$，$D=0$.

$$\overleftrightarrow{\Lambda}\cdot\vec{E}=\begin{pmatrix} S-n^2 & 0 & 0 \\ 0 & S-n^2 & 0 \\ 0 & 0 & P \end{pmatrix}\begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}=0 \Rightarrow \begin{cases} (S-n^2)E_x=0 \\ (S-n^2)E_y=0 \\ (P)E_z=0 \end{cases}$$

$\Rightarrow$ $n^2=1-\frac{\omega_{pe}^2}{\omega^2} \Rightarrow \omega^2=k^2c^2+\omega_{pe}^2$ ($E_x, E_y\neq0$). 横波模式 ($\vec{k}\perp\vec{E}$)，电磁横波. $\omega\geq\omega_{pe}$

$P=0 \Rightarrow \omega=\omega_{pe}$ ($E_z\neq0$)，纵波模式 ($\vec{k}\parallel\vec{E}$). ，静电波. Langmuir波

考虑热效应后：$\omega^2=\omega_{pe}^2+k^2C_{se}^2$

4.3.1 由公式导出考虑热效应时的静电Langmuir波

$$\frac{\partial n_e}{\partial t}+\nabla\cdot(n_e\vec{v}_e)=0$$

$$\rho_e\frac{d\vec{v}_e}{dt}=-\nabla p_e-en_e\vec{E}$$

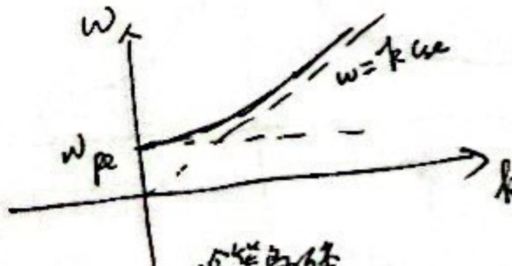
$$\nabla\cdot\vec{E}=\frac{e(n_0-n_e)}{\varepsilon_0}=-\frac{en_e'}{\varepsilon_0}$$

$\overset{FT}{\Longrightarrow}$ $\frac{n_e'}{n_{e0}}=\frac{\vec{k}\cdot\vec{v}_e}{\omega}$

$i\omega\rho_{e0}\vec{v}_e=i\vec{k}C_{se}^2m_en_e'+en_{e0}\vec{E} \Rightarrow i\omega\rho_{e0}\vec{k}\cdot\vec{v}_e=ik^2C_{se}^2m_en_e'+en_{e0}\vec{k}\cdot\vec{E}$

$i\vec{k}\cdot\vec{E}=-en_e'/\varepsilon_0$.

纵波，$\vec{k}\parallel\vec{E}\parallel\vec{v}_e$，回复力：热压力，电场力（静电、电荷分离），不满足准中性（$?: n_{e0}=n_{i0}$，$n_i'=0$）.



相速度 $\frac{\omega}{k}$，若 $k^2C_{se}^2\ll\omega_{pe}^2$，则 $\frac{\omega^2}{\omega_{pe}^2}=1+\frac{k^2C_{se}^2}{\omega_{pe}^2} \Rightarrow \frac{\omega}{\omega_{pe}}=1+\frac{k^2C_{se}^2}{2\omega_{pe}^2}$，$\omega_p=\omega_{pe}+\frac{k^2C_{se}^2}{2\omega_{pe}}$

$C_{se}^2=\gamma_e p_{e0}/\rho_{e0}$，$\gamma_e=3=\frac{2+D}{D}$（一维扰动）. $\frac{p_{e0}}{\rho_{e0}}=\frac{n_{e0}k_BT_e}{n_{e0}m_e}=V_{th}^2 \Rightarrow \omega_p=\omega_{pe}+\frac{3k^2V_{th}^2}{2\omega_{pe}}$

证明：冷等离子体 静电波 周期内 平均动能 与平均电场能量相等，二者之和为常值.

记：$\frac{\rho_{e0} v_e^2}{2}$，$\frac{\varepsilon_0 E^2}{2}$，由 $\vec{v}_e \simeq \vec{v}_e \exp(i\vec{k}\cdot\vec{r} - i\omega t) = \vec{v}_e \exp(ikz - i\omega t) = \vec{v}_e [\cos(kz-\omega t) + i\sin(kz-\omega t)]$ ⑦

只取实部，得动能密度为 $\frac{\rho_{e0}}{2} v_e^2 \cos^2(kz-\omega t)$，电场能量密度？

由 $i\omega\rho_{e0}\vec{v}_e = e n_{e0}\vec{E} \Rightarrow \vec{E} = \frac{i\omega m_e}{e}\vec{v}_e = \frac{i\omega m_e}{e}\vec{v}_e[\cos(kz-\omega t) + i\sin(kz-\omega t)]$，取实部得 $\vec{E} = -\frac{\omega m_e}{e}\vec{v}_e \sin(kz-\omega t)$

得 $\frac{\varepsilon_0 E^2}{2} = \frac{\varepsilon_0\omega^2 m_e^2}{2e^2} v_e^2 \sin^2(kz-\omega t)$，二者之和为：$\frac{\rho_{e0} v_e^2}{2}\cos^2(kz-\omega t) - \frac{\varepsilon_0\omega^2 m_e^2}{2e^2} v_e^2 \cos^2(kz-\omega t) + \frac{\omega^2 m_e^2 \varepsilon_0}{2e^2} v_e^2$

$$= \frac{\varepsilon_0\omega^2 m_e^2}{2e^2} v_e^2 = \frac{\rho_{e0} v_e^2}{2}\cdot\omega^2\frac{m_e\varepsilon_0}{n_{e0}e^2} = \frac{\rho_e v_e^2}{2}\frac{\omega^2}{\omega_{pe}^2} = \frac{\rho_e v_e^2}{2} = W_0$$

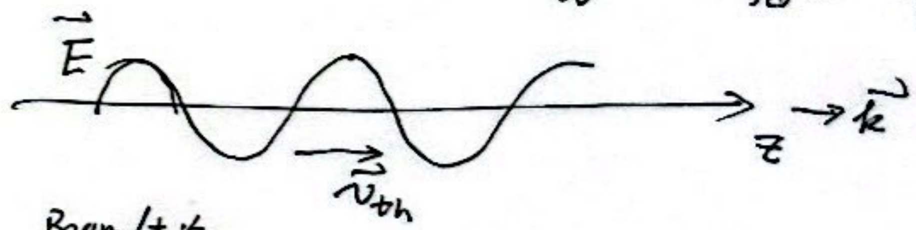
则 $\frac{\rho_{e0} v_e^2}{2} = W_0 \cos^2(kz-\omega t)$

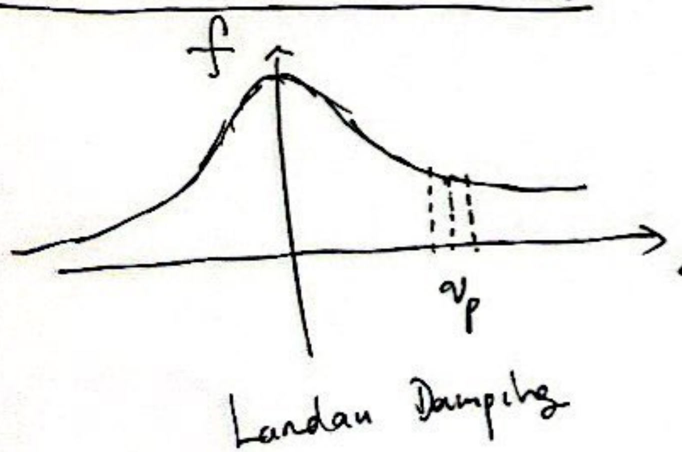
$\frac{\varepsilon_0 E^2}{2} = W_0 \sin^2(kz-\omega t)$，$\underline{\frac{\rho_{e0} v_e^2}{2} + \frac{\varepsilon_0 E^2}{2} = W_0}$．

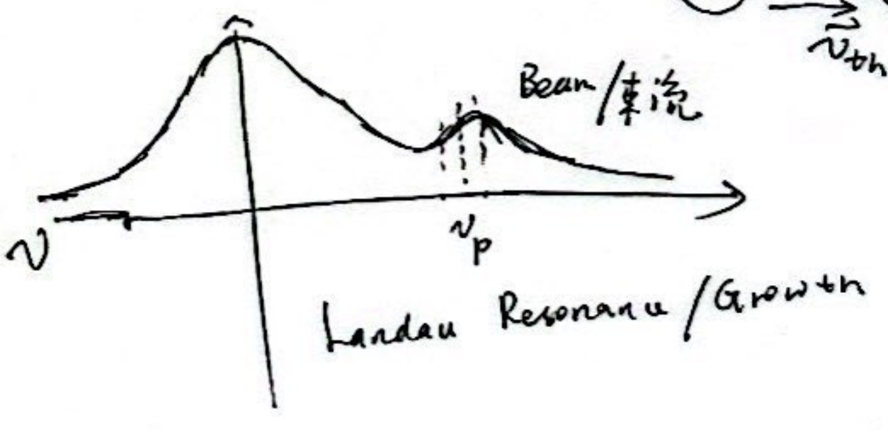
$\boxed{\text{证明}}$：考虑热效应后，有平均电场能量密度低于平均电子运动能量密度．$\left(\frac{\varepsilon_0 E^2}{2} \Big/ \frac{\rho_{e0} v_e^2}{2} = \frac{\omega_{pe}^2}{\omega^2} < 1 \right)$．

提示：$e n_{e0}\vec{E} = i\omega\rho_{e0}\vec{v}_e - i\vec{k} c_{se}^2 n_e e' = i\omega\rho_{e0}\vec{v}_e - i c_{se}^2 m_e n_{e0}\frac{\vec{k}(\vec{k}\cdot\vec{v}_e)}{\omega} = i\omega\rho_{e0}\vec{v}_e - \frac{i c_{se}^2 m_e n_{e0}}{\omega}(k^2\vec{v}_e)$

朗道阻尼 (Landau Damping)：$v_{th} \sim v_p = \omega/k$．







无碰撞阻尼
波粒共振
波的耗散与增长．
尾瘤 (Bump-on-tail Instab.)

4.3.2 非磁化等离体电磁模式． $\omega^2=\omega_{pe}^2+k^2c^2$．$(n^2=S)$ ⑧

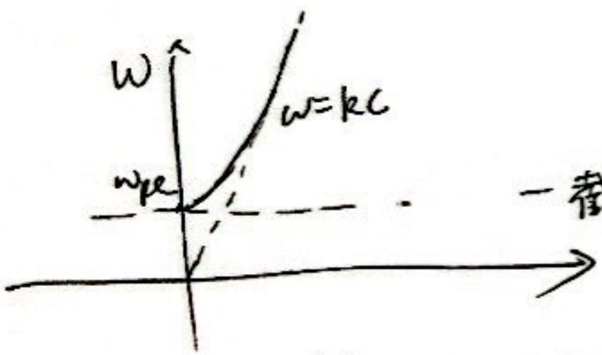
$$\left.\begin{aligned}&\frac{\partial\vec B}{\partial t}+\nabla\times\vec E=0\\&\nabla\times\vec B=\mu_0\vec j+\mu_0\varepsilon_0\frac{\partial\vec E}{\partial t},\quad \vec j=-en_e\vec v_e\\&\rho_e\frac{d\vec v_e}{dt}=-\nabla p_e-en_0\vec E\end{aligned}\right\}\Rightarrow\left\{\begin{aligned}&\vec k\times\vec E=\omega\vec B\\&i\vec k\times\vec B=-en_{e0}\mu_0\vec v_e-i\mu_0\varepsilon_0\omega\vec E=i\vec k\times(\vec k\times\vec E)/\omega\\&-i\omega n_{e0}m_e\vec v_e=-\cancel{i\vec k\rho_e C_{se}^2}-en_{e0}\vec E\end{aligned}\right.$$

$$i\vec k(\vec k\cdot\vec E)/\omega-i k^2\vec E/\omega = i(\vec k\vec k-k^2\vec I)\cdot\vec E/\omega$$

$$\Rightarrow \vec v_e=\frac{-ien_{e0}\vec E}{\omega n_{e0}m_e}$$

考虑电磁横波，($\vec k\cdot\vec E=0$，且) 可忽略去热压力梯度力作用(解耦)；得：$\dfrac{+ie^2n_0\mu_0}{\omega m_e}\vec E-i\dfrac{\omega}{c^2}\vec E=-\dfrac{ik^2\vec E}{\omega}$．

得：$\dfrac{\omega_{pe}^2}{c^2}-\dfrac{\omega^2}{c^2}+k^2=0\Rightarrow\omega^2=k^2c^2+\omega_{pe}^2$．

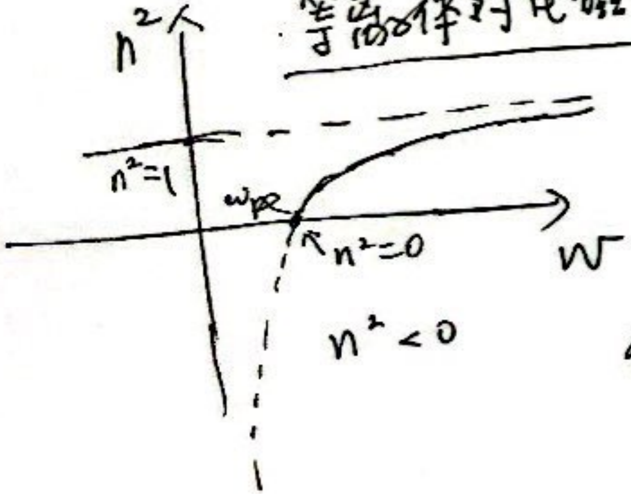


— 截止频率 (ω低于 ω_{pe}时，无法传播) $\left(n^2=\dfrac{k^2c^2}{\omega^2}=1-\dfrac{\omega_{pe}^2}{\omega^2}<1\right.$，折射率 $\left.n^2<0\right)$

$\omega>>\omega_{pe}$ 时．$\omega\to kc$ $(n^2\to1)$

趋近真空极限．

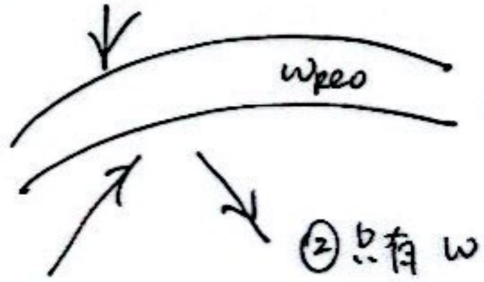
等离体对电磁光波的作用：(折射率小于1的值)．



$n^2>0$ 传播区．

$n^2<0$ 截止区

① 无线电波的跨洋传播 —— 电离层存在的证据．



电离层．(~60 – 1000 km)

② 只有 $\omega>\omega_{peo}$ 的信号才能通过电离层传向地面．③ 电离层探测 遥测卫料

$$\left\{\begin{array}{l} \text{电场能量密度 } \dfrac{\varepsilon_0 E^2}{2},\ \text{磁 } \dfrac{B^2}{2\mu_0},\quad \vec{k}\perp\vec{E} \Rightarrow \dfrac{E}{B}=\dfrac{\omega}{k} \Rightarrow \dfrac{\varepsilon_0E^2}{2}\Big/\dfrac{B^2}{2\mu_0}=\dfrac{\omega^2}{k^2c^2} \Rightarrow \text{若 } \omega^2>k^2c^2,\ \text{则 } \dfrac{\varepsilon_0E^2}{2}>\dfrac{B^2}{2\mu_0}, \\ \dfrac{\varepsilon_0E^2}{2}\Big/\dfrac{n_{e0}m_ev_e^2}{2} = \dfrac{\varepsilon_0E^2}{n_{e0}m_ev_e^2} = \dfrac{\varepsilon_0 \cancel{E^2}\,\omega^2 \cancel{n_{e0}^2} m_e^{\cancel{2}}}{n_{e0}\cancel{m_e}\, e^2 \cancel{n_{e0}^2}\, \cancel{E^2}} = \dfrac{\omega^2}{\omega_{pe}^2}.\quad \text{若 } \omega^2>\omega_{pe}^2,\ \text{则 } \dfrac{\varepsilon_0E^2}{2}>\dfrac{\rho_{e0}v_e^2}{2} \end{array}\right.$$

5.4 磁化等离子体静电模式 $(\vec{B}_0\neq0,\ \vec{B}'=0)$

$\vec{k}\times\vec{E}=\omega\vec{B}'=0$. $\vec{B}'=0$, $\vec{k}\parallel\vec{E}$. (纵波). $\vec{k}=k_x\hat{e}_x+k_z\hat{e}_z$, $\dfrac{k_x}{E_x}=\dfrac{k_z}{E_z}$.

$\omega\vec{k}\times\vec{B}=k\times(k\times\vec{E})=(\vec{k}\vec{k}-k^2\vec{I})\cdot\vec{E}=-\dfrac{\omega^2}{c^2}\vec{\varepsilon}\cdot\vec{E}=0$. (一般形式. 引入了更松的限制) $(\vec{B}=0,\ \vec{k}\times\vec{B}=0)$.

再用一个方程便可描述电场. $\vec{\varepsilon}\cdot\vec{E}\cdot\vec{k}=0$ (相当于将两个方程投影到k方向).

$$\vec{\varepsilon}\cdot\vec{E}=\begin{pmatrix} S & -iD & 0 \\ iD & S & 0 \\ 0 & 0 & P\end{pmatrix}\cdot\begin{pmatrix}E_x\\0\\E_z\end{pmatrix}=0 \Rightarrow \left\{\begin{array}{l} SE_x=0\\ iDE_x=0\\ PE_z=0\end{array}\right\},\ \text{由 } \vec{\varepsilon}\cdot\vec{E}\cdot\vec{k}=0 \Rightarrow \left.\begin{array}{l} SE_xk_x+PE_zk_z=0 \\ \Rightarrow \underbrace{P=-S\tan^2\theta}_{\text{一般形式}}\end{array}\right\}.$$

也可由方程组导出:

$$\left\{\begin{array}{l} \dfrac{\partial n_e'}{\partial t}+n_{e0}\nabla\cdot\vec{v}_e=0 \\ n_0m_e\dfrac{\partial\vec{v}_e}{\partial t}=-en_{e0}(\vec{E}+\vec{v}_e\times\vec{B}_0) \\ \nabla\times\vec{E}=0\quad(\vec{k}\parallel\vec{E}). \\ \mu_0\vec{j}+\dfrac{1}{c^2}\dfrac{\partial\vec{E}}{\partial t}=0,\quad \vec{j}=-en_{e0}\vec{v}_e\quad(\vec{v}_e\parallel\vec{E}). \\ \nabla\cdot\vec{E}=-\dfrac{en_e'}{\varepsilon_0} \end{array}\right. \Rightarrow$$

将速度用E_x表示:

$$-i\omega n_0m_ev_{ex}=-en_0(E_x+v_{ey}B_0)=-en_0\left(E_x+\frac{i\Omega_e v_{ex}}{\omega}B_0\right)$$

$$v_{ey}=\frac{iev_{ex}B_0}{\omega m_e}=i\frac{\Omega_e}{\omega}v_{ex}$$

$$-i\omega n_0m_ev_{ez}=-en_0E_z=-en_0\frac{k_z}{k_x}E_x$$

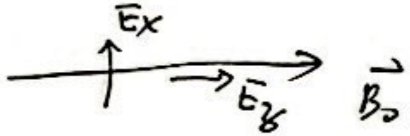
$$\Rightarrow\left\{\begin{array}{l} v_{ex}=-eE_x\Big/m_e\left(-i\omega+\dfrac{i\Omega_e^2}{\omega}\right) \\ v_{ez}=\dfrac{e}{i\omega m_e}\dfrac{k_z}{k_x}E_x \end{array}\right.$$

(19)

再由安培定律或泊松方程 $i(k_x E_x + k_z E_z) = -\frac{e n_e'}{\varepsilon_0} = -\frac{e n_0}{\omega \varepsilon_0}(k_x v_{ex} + k_z v_{ez})$，代入可得 $\begin{cases} 1-\frac{\omega_{pe}^2}{\omega^2} = -\left(1-\frac{\omega_{pe}^2}{\omega^2-\Omega_{ce}^2}\right)(\tan^2\theta) \\ \tan^2\theta = \frac{k_x^2}{k_z^2}. \end{cases}$

$\theta = 0^\circ$，$P=0$，Langmuir 波 $(E_z \neq 0, E_x = 0)$

$\theta = 90^\circ$，$S = 0$．$(E_x \neq 0, E_z = 0)$



$\omega^2 = \omega_{pe}^2 + \Omega_{ce}^2$．高杂频率．

$1-\frac{\omega_{pe}^2}{\omega^2-\Omega_{ce}^2} \sim 1-\frac{\omega_{pe}^2}{\omega^2}\left(1+\frac{\Omega_{ce}^2}{\omega^2}\right)$

当 $\omega^2 >> \omega_{ce}^2$ 时，可化简一般情况色散关系：$\underline{\omega^2 = \omega_{pe}^2 + \omega_{ce}^2 \sin^2\theta}$，$\theta$ 为 $\vec{k}, \vec{B_0}$ 夹角．

考虑热效应时，有 $\underline{\omega^2 = \omega_{uH}^2 + k^2 C_{se}^2}$．(垂直传播时，Lorentz 力也参与进来，使得波动可以更快地回复！

4.5 磁化等离子体电磁模式． $\underline{\left(\tan^2\theta = -\frac{P(n^2-R)(n^2-L)}{(n^2-P)(Sn^2-RL)}\right.}$ ：Allis 形式）．

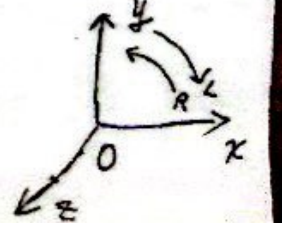
平行 $(\vec{k} \parallel \vec{B_0})$ ：$\underline{n^2 = R,\ n^2 = L}$；垂直 $(\vec{k} \perp \vec{B_0})$ ：$\underline{n^2 = P,\ n^2 = RL/S}$

由电场方程：$\underset{\substack{\downarrow \\ (3\times3)\text{色散矩阵}}}{\overset{\leftrightarrow}{\Lambda}} \cdot \vec{E} = \begin{pmatrix} S-n^2\cos^2\theta & -iD & n^2\sin\theta\cos\theta \\ iD & S-n^2 & 0 \\ n^2\sin\theta\cos\theta & 0 & P-n^2\sin^2\theta \end{pmatrix}\begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} \overset{\theta=0}{=\!=} \begin{pmatrix} S-n^2 & -iD & 0 \\ iD & S-n^2 & 0 \\ 0 & 0 & P \end{pmatrix}\begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}$

$$\overset{\theta=0}{\Longrightarrow} \begin{cases} (S-n^2)E_x - iDE_y = 0 \Rightarrow \frac{E_x}{E_y} = \frac{iD}{S-n^2} \\ iDE_x + (S-n^2)E_y = 0 \Rightarrow \frac{E_x}{E_y} = -\frac{S-n^2}{iD} \\ PE_z = 0 \Rightarrow P = 0\ (E_z \neq 0)\text{，静电 Langmuir 波，不再讨论。} \end{cases}$$

$\Rightarrow (S-n^2)^2 = D^2 \Rightarrow \begin{cases} ① S-n^2 = \pm D \Rightarrow \begin{cases} n^2 = R \\ n^2 = L \end{cases} \\ ② \frac{E_x}{E_y} = \pm i \end{cases}$

而 $\left|\frac{E_x}{E_y}\right| = 1 \Rightarrow$ 圆偏振，$E_x = iE_y$ 对应于 ___ 旋圆，$E_x = -iE_y$ 对应于 ___ 旋？



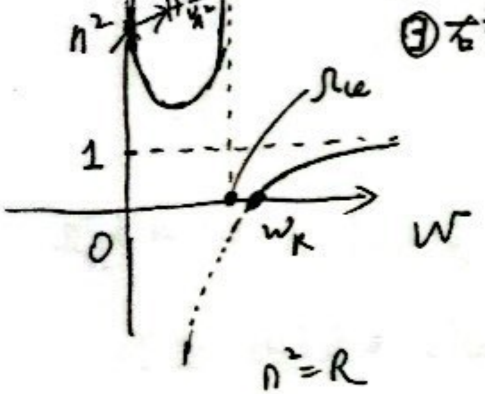
"左右旋"简单判定： $\vec{E_x} = \bar{E}_x e^{i(\vec{k}\cdot\vec{r}-\omega t)} = \bar{E}_x \cos(\vec{k}\cdot\vec{r}-\omega t) + i\bar{E}_x \sin(\vec{k}\cdot\vec{r}-\omega t) = \bar{E}_x(\cos\varphi + i\sin\varphi)$ ⑩

$E_y = -iE_x$（$E_x = iE_y$）时，$\bar{E}_y = \bar{E}_x(-i\cos\varphi + \sin\varphi)$

t_1 时刻（$\varphi = 0$）时，$\varphi = 0$，得：$\bar{E}_x \sim \bar{E}_x$，$E_y \simeq 0$；$t_2(>t_1)$ 时刻，（$\varphi = -\frac{\pi}{2}$）时，$\bar{E}_x \sim 0$，$E_y \simeq -\bar{E}_x$

右旋：$n^2 = R = 1 - \frac{\omega_{pe}^2}{\omega(\omega+\omega_{ce})}$ ①由 $R = 0$，得 $\omega_R = \frac{\Omega_{ce}}{2} + \sqrt{\omega_{pe}^2 + \frac{\Omega_{ce}^2}{4}}$，右旋截止频率

$= 1 - \frac{\omega_{pe}^2}{\omega(\omega - \Omega_{ce})}$ ②由 $n^2 \to \infty$ 得 $\omega \sim \Omega_{ce}$，对应于（右旋）电子回旋共振



$n^2 = R$

③右旋波低频极限

$\omega \to 0$ 时，需考虑离子效应，$n^2 = 1 - \sum_\alpha \frac{\omega_{p\alpha}^2}{\omega(\omega + \omega_{c\alpha})}$

$n^2 = 1 + \frac{c^2}{v_A^2}$

$= 1 - \frac{\omega_{pe}^2}{\omega(\omega - \Omega_{ce})} - \frac{\omega_{pi}^2}{\omega(\omega + \Omega_{ci})}$

$\frac{k^2c^2}{\omega^2} \simeq \frac{c^2}{v_A^2} \Rightarrow \omega^2 = k^2 v_A^2$

$\sim 1 + \frac{\omega_{pe}^2}{\omega\Omega_{ce}} - \frac{\omega_{pi}^2}{\omega\Omega_{ci}}$

（低频）右旋剪切 Alfven 波

④高频极限：$n^2 = 1$（$\omega^2 = k^2c^2$）

$= 1 + \frac{1}{\omega}\frac{\omega_{pe}^2\Omega_{ci} - \omega_{pi}^2\Omega_{ce}}{\Omega_{ce}\Omega_{ci}} = 1 + \frac{1}{\omega} \to \infty$

并不合适，忽略得太彻底了！

存在两组右旋波解：$\omega \geq \omega_R$ 时，右旋高频电磁波

$\omega < \Omega_{ce}$ 时，电子回旋波 Electron Cyclotron Wave［哨声波（Whistler mode）］

$n^2 \simeq 1 + \frac{\omega_{pe}^2}{\omega\Omega_{ce}}(1 + \frac{\omega}{\Omega_{ce}}) - \frac{\omega_{pi}^2}{\omega\Omega_{ci}}(1 - \frac{\omega}{\Omega_{ci}})$

$\Omega_{ce} < \omega \leq \omega_R$ 时，无解（截止区），$n^2 < 0$

$= 1 + \frac{\omega_{pe}^2}{\Omega_{ce}^2} + \frac{\omega_{pi}^2}{\Omega_{ci}^2} = 1 + c^2(\frac{\mu_0\rho_0}{B_0^2}) = \frac{c^2}{v_A^2}$

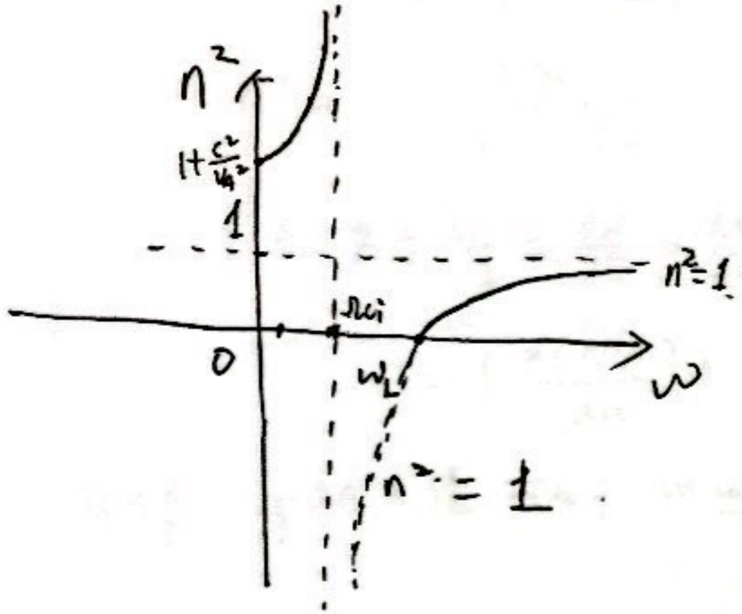
无背景场时，$\vec{B_0} = 0$ 时：$n^2 = 1 - \frac{\omega_p^2}{\omega^2}$ → 左、右旋差异消失。

$(\frac{n_0 e^2}{m_e \varepsilon_0} \cdot \frac{m_e^2}{e^2 B_0^2})$

分析：为何 $n^2 - \omega$ 在 $\omega < \Omega_{ce}$ 区间出现极小值（v_{pmax}）与非单调变化？

$= 1 + c^2/v_A^2$

左旋：$n^2 = L = 1 - \frac{\omega_{pe}^2}{\omega(\omega+\Omega_{ce})}$， ① 由 $n^2=0$，得 $\omega_L = -\frac{\Omega_{ce}}{2} + \sqrt{\omega_{pe}^2 + \frac{\Omega_{ce}^2}{4}}$，左旋截止频率 (11)



② 由 $n^2 \to \infty$，得 需考虑离子效应 $n^2 = L = 1 - \frac{\omega_{pi}^2}{\omega(\omega-\Omega_{ci})} - \frac{\omega_{pe}^2}{\omega(\omega+\Omega_{ce})}$

$\omega \to \Omega_{ci}$，离子回旋共振（左旋离子回旋波）

③ 低频极限． $n^2 = 1 + \frac{c^2}{V_A^2}$

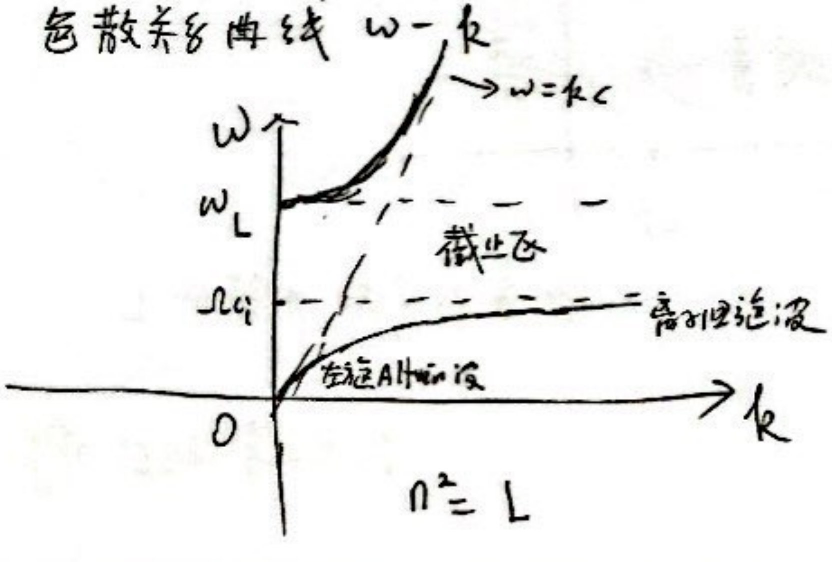
$\omega^2 = k^2 V_A^2$．（低频）左旋剪切 Alfvén 波．

④ 无磁场．真空极限． Okay．

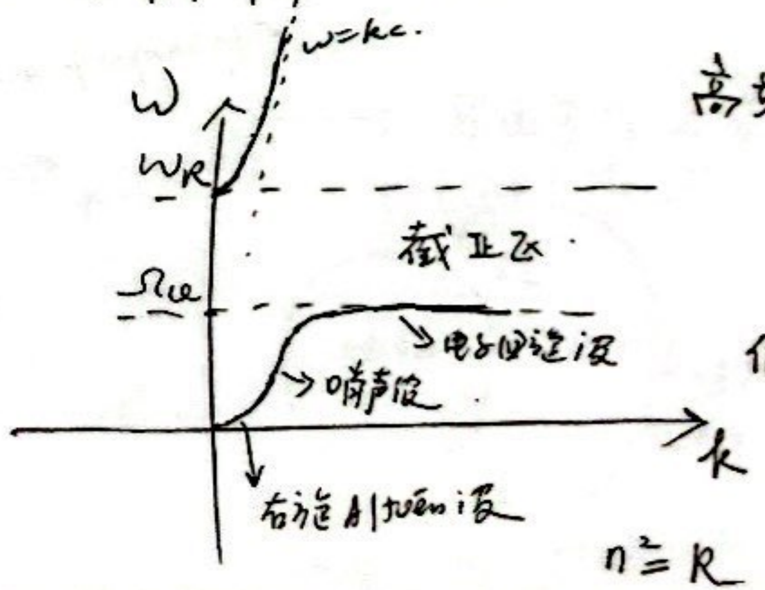
$\Omega_{ci} < \omega < \omega_L$：左旋波截止区．，

作业：比较 Ω_{ce}．ω_{LH}，ω_{UH}，ω_L，ω_R，ω_{pe} 大小关系．

色散关系曲线 $\omega - k$



$n^2 = L$



高频传播区

低频传播区．

$n^2 = R$

几点：① 电子回旋共振、离子回旋共振 ⟶ 波粒共振相互作用，

② 哨声波：ω 略低于 Ω_{ce}，why whistler？ n^2-ω 非寻常性。

$$n^2 = 1 - \frac{\omega_{pe}^2}{\omega(\omega - \Omega_{ce})}，\quad 设 v_p = \frac{\omega}{k} \ll c，得 \quad n \sim \frac{\omega_{pe}}{[(\Omega_{ce}-\omega)\omega]^{1/2}}，\quad \frac{v_p}{c} \sim \frac{[(\Omega_{ce}-\omega)\omega]^{1/2}}{\omega_{pe}}$$

相速度有极值，$\omega \sim \frac{\Omega_{ce}}{2}$，$v_{p\max} = \frac{\Omega_{ce} \cdot c}{2\omega_{pe}}$

得群速度 $v_g = \frac{d\omega}{dk} = \left(\frac{dk}{d\omega}\right)^{-1}$

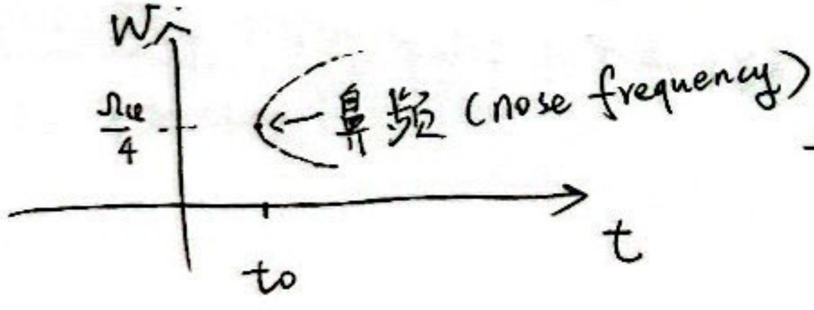
$$= \left(\frac{d(n\omega/c)}{d\omega}\right)^{-1} = \frac{c}{\omega\frac{dn}{d\omega} + n} \simeq \frac{2c}{\omega_{pe}\Omega_{ce}} \omega^{1/2} (\Omega_{ce} - \omega)^{3/2}$$

[自行推导、验证]。

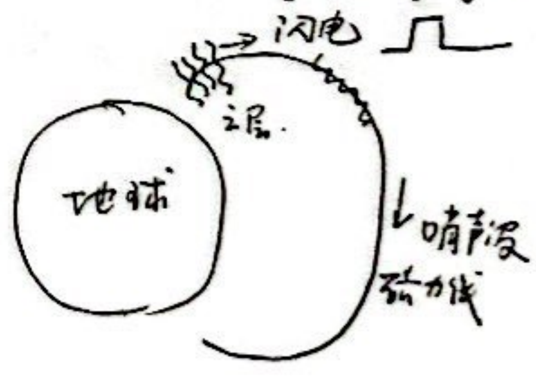
可得 v_g 极大值对应于 $\omega \simeq \Omega_{ce}/4$，即 $\omega \leq \frac{\Omega_{ce}}{4}$ 时，$v_g(\omega)$ 随 $\omega\uparrow$ 而 $\uparrow$，

$\omega \geq \frac{\Omega_{ce}}{4}$ 时，$v_g(\omega)$ 随 $\omega\uparrow$ 而 $\downarrow$。

某处台站，接收空间电磁信号（几 + KHz）直接转为音频 —— 哨声（$\omega \leq \frac{\Omega_{ce}}{4}$，高频先抵达）。



⟶ 可用于空间等离子体参数诊断。



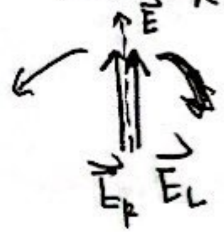
上网聆听哨声波音频。

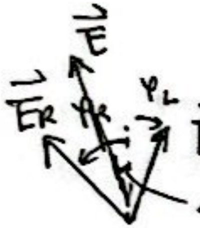
③ 法拉第旋转

(2-1)

线偏振 $\longleftrightarrow$ 左、右旋圆偏振（同 ω），在传播过程中，$n_R \neq n_L$，$\vec{E}$ 发生旋转。

$\vec{E} = \vec{E}_L + \vec{E}_R$





$\Delta\varphi = \frac{\varphi_R - \varphi_L}{2} \left(= \varphi_R - \frac{\varphi_R + \varphi_L}{2} \right)$

$$\begin{cases} \vec{E}_L = \frac{E_0}{\sqrt{2}} (\vec{e}_x - i\vec{e}_y) \exp(i(\vec{k}_L \cdot \vec{r} - \omega t)) \\ \vec{E}_R = \frac{E_0}{\sqrt{2}} (\hat{e}_x + i\hat{e}_y) \exp(i(\vec{k}_R \cdot \vec{r} - \omega t)) \end{cases}$$

φ_L

φ_R

$\Delta\varphi = \left| \frac{\varphi_R - \varphi_L}{2} \right| = \frac{(k_R - k_L) z}{2}$，设 $\boxed{\begin{matrix} \omega \gg \Omega_{ce} \\ \omega \gg \omega_{pe} \end{matrix}}$，得：$n_R = \frac{k_R c}{\omega} \sim \sqrt{1 - \frac{\omega_{pe}^2}{\omega(\omega - \Omega_{ce})}}$

$\sim 1 - \frac{\omega_{pe}^2}{2\omega(\omega - \Omega_{ce})}$

$\therefore \Delta\varphi = \frac{\omega z}{c^2} \left(\frac{\omega_{pe}^2}{2\omega(\omega + \Omega_{ce})} - \frac{\omega_{pe}^2}{2\omega(\omega - \Omega_{ce})} \right)$

$n_L = \frac{k_L c}{\omega} \sim 1 - \frac{\omega_{pe}^2}{2\omega(\omega + \Omega_{ce})}$

$|\Delta\varphi| = \frac{\omega z}{2c} \frac{\omega_{pe}^2 \, 2\Omega_{ce}}{2\omega(\omega^2 - \Omega_{ce}^2)} \sim \frac{z \omega_{pe}^2 \Omega_{ce}}{2c\omega^2} \propto (n_0, z, B_0, \omega)$

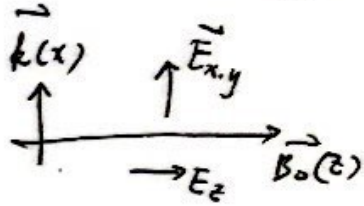
则可 ~~基于~~ 在电磁波穿越等离子体时，$\Delta\varphi$ 测量，推断等离子体参数性质。① 天文射电源、行星际介质、相位变化

IPS

② 等离子体诊断

4.5.2 垂直传播： $n^2=P$， $n^2=RL/S$.

$$\overleftrightarrow{\Lambda}\cdot\vec{E}=\begin{pmatrix} S & -iD & 0\\ iD & S-n^2 & 0\\ 0 & 0 & P-n^2\end{pmatrix}\cdot\begin{pmatrix}E_x\\E_y\\E_z\end{pmatrix}=0 \Rightarrow \begin{cases} SE_x - iDE_y=0\\ iDE_x+(S-n^2)E_y=0\\ (P-n^2)E_z=0\end{cases}$$



1° $P=n^2 \Rightarrow \omega^2=k^2c^2+\omega_{pe}^2$， $E_z\neq 0$，平行 $\vec{B_0}$ 的电场扰动，与无磁场情形相同. $\longrightarrow$ Ordinary Mode

2° $n^2=RL/S$， $SE_x=iDE_y$， $\left(\frac{iD}{S}\right)^2+(S-n^2)=0 \Rightarrow n^2=RL/S$， $E_x=\frac{iD}{S}E_y$， $\begin{cases}\frac{D}{S}>0\text{时，左旋椭圆偏振}\\ \frac{D}{S}<0\text{时. 右}\cdots\end{cases}$

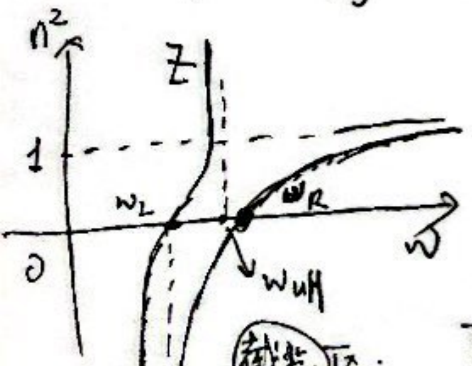
$$\frac{D}{S}=\frac{\omega_{pe}^2\omega_{ce}}{\omega(\omega^2-\Omega_{ce}^2)}\Big/1-\frac{\omega_{pe}^2}{\omega^2-\omega_{ce}^2}=\frac{\omega_{pe}^2\,\omega_{ce}}{(\omega^2-\omega_{UH}^2)\,\omega}$$

$$\rightarrow n^2=\frac{\left(1-\frac{\omega_{pe}^2}{\omega(\omega-\Omega_{ce})}\right)\left(1-\frac{\omega_{pe}^2}{\omega(\omega+\Omega_{ce})}\right)}{1-\frac{\omega_{pe}^2}{\omega^2-\Omega_{ce}^2}}=1-\frac{\omega_{pe}^2}{\omega^2}\frac{\omega^2-\omega_{pe}^2}{\omega^2-\omega_{UH}^2}$$

可化为 $\frac{D}{S}=\frac{\omega_{pe}^2(\omega^2-\omega_{pe}^2)}{(\omega^2-\omega_{UH}^2)\omega^2}\frac{(-\omega\Omega_{ce})}{\omega^2-\omega_{pe}^2}$

$\frac{D}{S}=(n^2-1)\frac{\omega\Omega_{ce}}{\omega^2-\omega_{pe}^2}$，即当 $\begin{cases}① \ (n^2-1)(\omega^2-\omega_{pe}^2)>0\text{时，为左旋} \begin{cases}n^2>1,\ \omega^2>\omega_{pe}^2\\ n^2<1,\ \omega^2<\omega_{pe}^2\end{cases}\\ ② \ (n^2-1)(\omega^2-\omega_{pe}^2)<0\text{时，为右旋} \begin{cases}n^2>1,\ \omega^2<\omega_{pe}^2\\ n^2<1,\ \omega^2>\omega_{pe}^2\end{cases}\end{cases}$

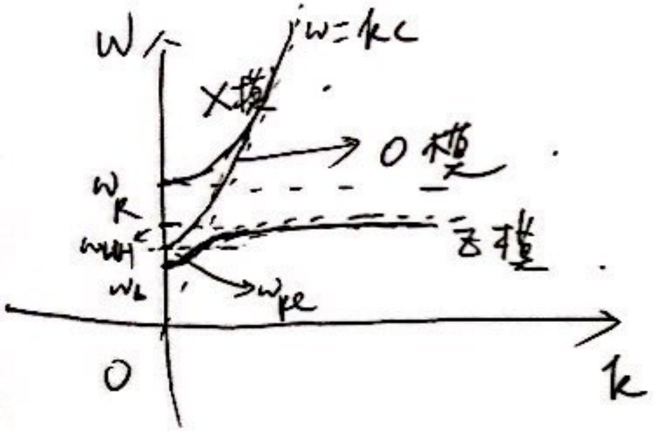
$n^2=RL/S=0$. 截止频率 $\begin{cases}R=0,\ \omega_R \longrightarrow \text{高频传播区的波的截止频率（X模）}\\ L=0,\ \omega_L \rightarrow \text{Z模的截止频率.}\end{cases}$



截止区.

Z模共振：$n^2=\frac{RL}{S}\rightarrow\infty$， $S=0 \Rightarrow 1=\frac{\omega_{pe}^2}{\omega^2-\omega_{ce}^2} \Rightarrow \omega^2=\omega_{ce}^2+\omega_{pe}^2=\omega_{UH}^2$ 高杂共振

$E_x=\frac{iD}{S}E_y$，故共振时有 $E_x\gg E_y \Rightarrow \vec{k}\parallel\vec{E} \Rightarrow$ 静电（共振时趋于静电振荡）



X模与O模的差异是磁场 $\vec{B}_0$ 造成的，$\vec{B}_0=0$ 时，$\omega^2=k^2c^2+\omega_{pe}^2$

右旋

或 $n^2=1-\frac{\omega_{pe}^2}{\omega^2}$

高频极限 $\omega=kc$.

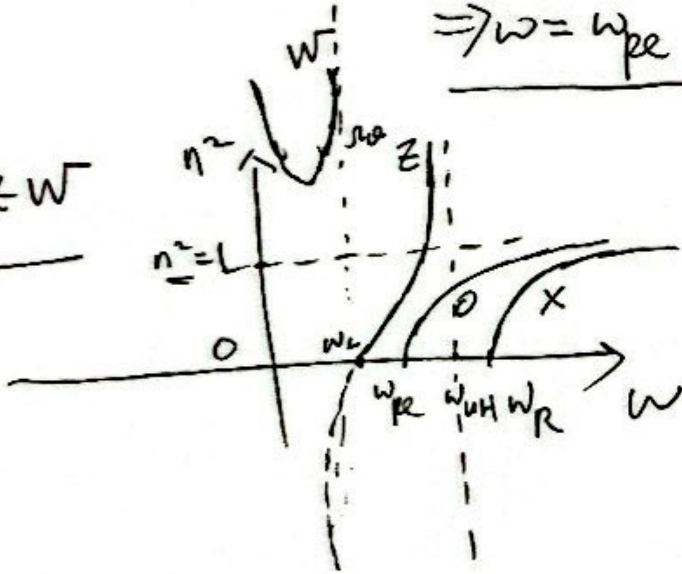
对于Z模，$n^2=1$ 时 $RL=S$，算得：

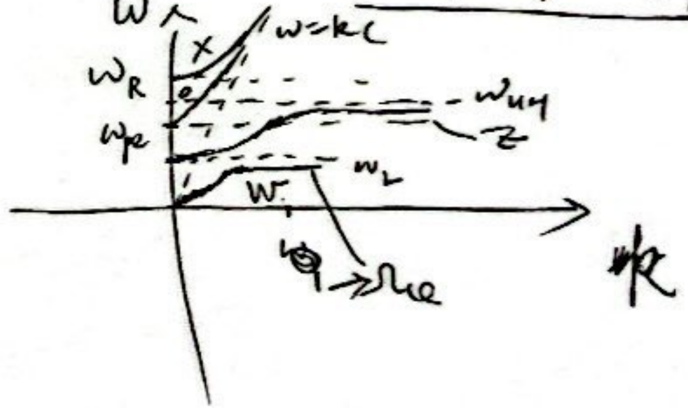
$$\left(1-\frac{\omega_{pe}^2}{\omega(\omega-\Omega_{ce})}\right)\left(1-\frac{\omega_{pe}^2}{\omega(\omega+\Omega_{ce})}\right)=\frac{(\omega^2-\omega\Omega_{ce}-\omega_{pe}^2)(\omega^2+\omega\Omega_{ce}-\omega_{pe}^2)}{\omega^2(\omega^2-\Omega_{ce}^2)}=1-\frac{\omega_{pe}^2}{\omega^2-\omega_{ce}^2}$$

$\Rightarrow \omega=\omega_{pe}$. 则对于Z模：$\left.\begin{array}{l}\omega>\omega_{pe}\text{时}, n^2>1\\ \omega<\omega_{pe}\text{时}, n^2<1\end{array}\right\}$ 左旋.

一般情况，X O Z W

磁离子理论.





4.6 低频近似静电波 ($\omega \ll \Omega_{ce}, \omega_{pe}$，设 $v_p \ll c_{se}$，即相速远小于电子热速，可略去电子惯性项) 15

4.6.1 非磁化情况 ($\vec{B}_0=0$, $\vec{B}'=0$)：离子声波与离子 Langmuir 波.

$$\begin{cases} \dfrac{\partial n_i'}{\partial t} + n_0 \nabla\cdot\vec{v}_i = 0, \quad \dfrac{\partial n_e'}{\partial t} + n_0\nabla\cdot\vec{v}_e = 0 \Rightarrow \omega n_e' \sim n_0 k v_e \\ -\nabla p_e' - e n_0 \vec{E} = 0 \\ n_0 m_i \dfrac{\partial \vec{v}_i}{\partial t} = -\nabla p_i' + e n_0 \vec{E} \\ \nabla\cdot\vec{E} = e(n_i' - n_e')/\varepsilon_0 \end{cases}$$

($v_p \ll c_{se}$，可略去 $n_0 m_e \dfrac{\partial \vec{v}_e}{\partial t} \sim \omega n_0 m_e \vec{v}_e$，$-\nabla p_e' \sim k n_e' m_e c_{se}^2$，$|\omega| \ll \left|\dfrac{k^2}{\omega}\right|$

$v_p^2 = \dfrac{\omega^2}{k^2} \ll c_{se}^2$

$$\Rightarrow \begin{cases} \dfrac{n_i'}{n_0} = \dfrac{k v_i}{\omega} \\ n_e' = \dfrac{i e n_0 E}{k \gamma_e k_B T_e} = \dfrac{i e n_0 E}{k c_{se}^2 m_e} \\ n_i' = \dfrac{i e n_0 E}{\omega^2 m_i/k - k c_{si}^2 m_i} \\ i k E = \dfrac{e(n_i' - n_e')}{\varepsilon_0} \end{cases} \Rightarrow 1 = \frac{\omega_{pi}^2}{\omega^2 - k^2 c_{si}^2} - \frac{\omega_{pe}^2}{k^2 c_{se}^2} \Rightarrow \omega^2 = k^2 \Big[\underbrace{c_{si}^2 + \frac{c_{se}^2 \omega_{pi}^2}{\omega_{pe}^2 + k^2 c_{se}^2}}_{v_s^2\ (\text{离子声速})} \Big]$$

且有 $\dfrac{n_e'}{n_i'} = \dfrac{\omega^2 - k^2 c_{si}^2}{k^2 c_{se}^2 m_e/m_i} = \dfrac{1}{1+\gamma_e k^2 \lambda_{De}^2}$

$$= k^2 \left[c_{si}^2 + \frac{\gamma_e k_B T_e}{m_i (1 + \gamma_e k^2 \lambda_{De}^2)} \right] \quad \left(\lambda_{De}^2 = v_{the}^2 / \omega_{pe}^2 \right).$$

1) 长波近似，$\lambda \gg \lambda_{De}$，得 $k\lambda_{De} \ll 1$，$n_e'/n_i' = 1$，准中性条件满足.

得 $\omega^2 = k^2 \Big(\underbrace{c_{si}^2 + \gamma_e k_B T_e / m_i}_{} \Big)$. 电子热压力通过电场传至离子上。

$$v_s^2 = c_{si}^2 + c_{se}^2 \frac{m_e}{m_i}$$

离子温度 $T_i \sim 0$ 时，仍有离子声波 (Ion Acoustic Mode)

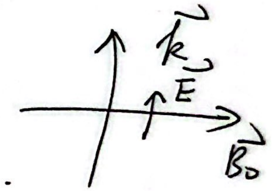
2) 短波近似，$\lambda << \lambda_{De}$，得 $\omega^2 = k^2 (C_{si}^2 + \frac{\gamma_e k_B T_e / m_i \ \omega_{pe}^2}{\gamma_e k^2 k_B T_e / m_e}) = \underline{k^2 C_{si}^2 + \omega_{pi}^2}$．离子 Langmuir 波．

$\frac{n_e'}{n_e'} \sim \gamma_e k^2 \lambda_{De}^2 >> 1$，此时~~电中性~~$\boxed{\text{无法维持}}$（Why？），$\underbrace{\text{电子不动}}_{\text{均匀背景}}$，离子静电振荡 +~~热压~~ 离子声波的 Landau 阻尼：$v_s \sim v_{thi} = \sqrt{k_B T_i / m_i}$，当 $T_e \sim T_i$ 时，强阻尼；当 $T_e >> T_i$ 时，阻尼较弱．

4.6.2 磁化情况（$\vec{B_0} \neq 0$，$\vec{B}' = 0$）：低杂静电波与静电离子回旋波（Lower Hybrid Mode）（Electrostatic Ion Cyclotron mode）

只讨论长波情况，无电荷分离（$n_i' = n_e'$）

1) 平行传播（$\vec{k} \parallel \vec{B_0}$）时，磁场不起作用，与非磁化情况相同．



2) 垂直传播或准垂直传播情况（$\vec{k} \perp \vec{B_0}$），$k \perp B_0$．$\vec{k} = k \hat{e}_x$，则 $\vec{E} = E_x \hat{e}_x$．

可知 Lorentz 力将影响波的振荡，有关线性化．FT 后的方程组为：

$$\left.\begin{aligned} \frac{n_e'}{n_0} &= \frac{k_x v_{ex}}{\omega} \\ -i\omega n_0 m_e v_{ex} &= -i k_x C_{se}^2 m_e n_e' - e n_0 E_x - e n_0 B_0 v_{ey} \\ -i\omega n_0 m_e v_{ey} &= e n_0 v_{ex} B_0 \end{aligned}\right\} \Rightarrow v_{ex} \left(1 - \frac{\Omega_{ce}^2}{\omega^2} - \frac{k_x^2 C_{se}^2}{\omega^2}\right) = -i \frac{e E_x}{m_e \omega}$$

同样得：$v_{ix} \left(1 - \frac{\Omega_{ci}^2}{\omega^2} - \frac{k_x^2 C_{si}^2}{\omega^2}\right) = \frac{i e E_x}{m_i \omega}$

由 $n_i' = n_e'$ 得 $v_{ex} = v_{ix}$，得：$\omega^2 = \underbrace{\Omega_{ce} \Omega_{ci}}_{\omega_{LH}^2} + k_x^2 v_s^2$，$v_s^2 = C_{si}^2 + \frac{\gamma_e k_B T_e}{m_i}$，$\underline{\text{低杂静电波}}$

ω_{LH}^2 低杂频率，如何理解这一低杂频率的出现/意义？

低杂频率的物理意义： $\omega_{LH} \gg \Omega_{ci}$， $\omega_{LH} \ll \Omega_{ce}$，故该静电振荡对于离子与电子"快""慢"不同。 (17)

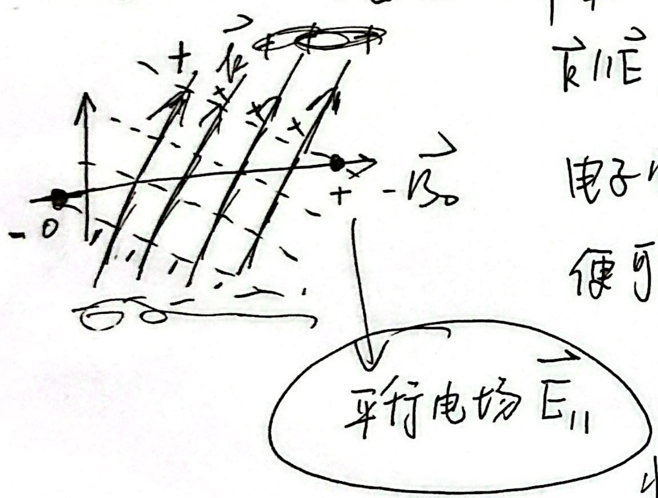
对于离子，快，故离子在波场中主要表现为随E的快速振荡： $m_i \frac{d^2x_i}{dt^2} = eE \Rightarrow |m_i \omega^2 x_i| = |eE|$

对于电子，慢，故电— — — — — — — — — 在缓变电场E作用下的

相似漂移运动： $\frac{dx_e}{dt} = \frac{m_e}{eB^2}\frac{dE}{dt} \sim \left|\omega \frac{m_e}{eB^2}E\right| = |\omega x_e|$

准中性要求 $x_i = x_e$，得： $\frac{m_e}{eB^2} = \frac{e}{m_i\omega^2} \Rightarrow \omega^2 = \Omega_{ce}\Omega_{ci} = \omega_{LH}^2$，即电子在垂直x方向的相似漂移要由离子在电场驱动下的相同的振荡

当 $\vec{B_0}$ 与 $\vec{k}$ 准垂直时，不需要如此"中和"方式： ~~电子~~



$\vec{k} \parallel \vec{E}$，则电场有平行于 $\vec{B_0}$ 的分量（$\vec{E_{\parallel}}$），在 $\vec{E_{\parallel}}$ 作用下

电子将做快速往返运动，（$\vec{k}$方向运动分量）可中和离子静电扰动 (v_{ix}, n_i')，此时仍使用x方向电子运动 方程

便可忽略电子惯性项回旋运动项： $0 = -ik_x C_{se}^2 m_e n_e' - e n_0 E_x$ ~~$en_0B_0v_{ey}$~~ ?

（设 $v_{the} \gg v_p$）. $\Rightarrow E_x = \frac{-ik_x C_{se}^2 m_e n_e'}{e n_0}$

将Ex代入 $v_{ix}\left(1 - \frac{\Omega_{ci}^2}{\omega^2} - \frac{k^2C_{si}^2}{\omega^2}\right) = \frac{k_x C_{se}^2 m_e n_i'}{m_i n_0 \omega}$， $\frac{n_i'}{n_0} = \frac{k_x v_i}{\omega}$

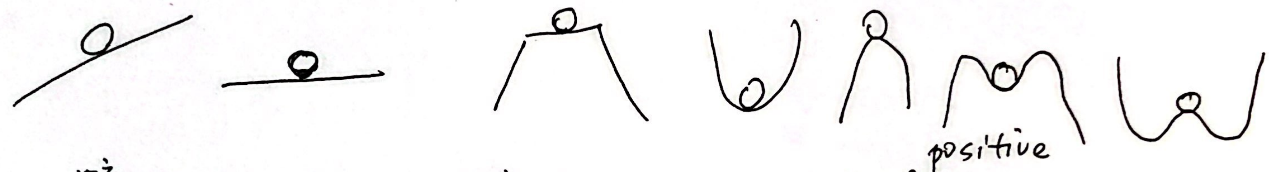
$\omega^2 - \Omega_{ci}^2 - k^2C_{si}^2 = \frac{k_x^2 C_{se}^2 m_e}{m_i} \Rightarrow \omega^2 \approx \Omega_{ci}^2 + k^2 v_s^2$

（长波）	静电离子回旋波 与 低杂波 在维持电中性方面
电中性	具有不同的物理： 准垂直下的平行电场驱动电子快速运动
$n_i' = n_e'$	与 垂直情况下的缓变电场电子漂移.
$v_{ix} = v_{ex}$	

第5章 等离子体不稳定性简介

5.1 基本概念与研究方法简介

平衡 (Equilibrium) & 稳定性 (Stability)：



平衡 → 受力状态. 稳定性 → 演化情况.

不稳定性 (Instability)：受到扰动时，相应力的作用会使扰动幅度增大 —— 正反馈 (positive feedback)

自由能 (free energy)

种类：
- 宏观不稳定 (MHD. 空间分布不均匀等)、微观不稳定 (速度分布函数偏离平衡态).
- 腊肠 (sausage)、扭曲 (kink)、火龙带 (firehose)、气球模 (balloon).
- 电磁、静电. / RT (Rayleigh-Taylor). KH (Kelvin-Helmholtz)

Streaming Instability / Tearing-mode Instability.

分析方法：直观分析 (定性)

能量原理 (MHD系统).

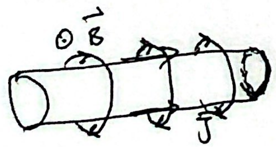
色散关系：扰动幅度随时间增长，小扰动，线性化 (仅适用于线性增长阶段)

$$A(t)\exp(-i(\omega t-\vec{k}\cdot\vec{r})),\quad A(t)=A_0e^{\gamma t}=A_0e^{-i\cdot i\gamma t}，得 A(t)=A_0e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

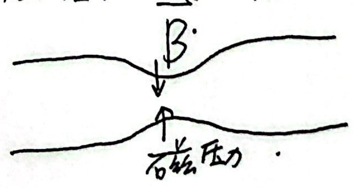
$$令\ \omega=\omega_r+i\gamma.\quad A(t)=A_0e^{\gamma t}e^{i(\vec{k}\cdot\vec{r}-\omega_r t)}$$

，故波动分析的FT方法仍可适用！

5.2 磁约束等离子柱的腊肠与扭曲模不稳定性

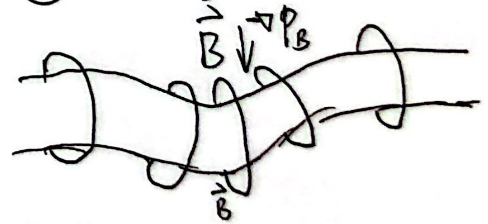


① 某处内缩，$B\uparrow$，使内缩之继续，如何致稳？



② 某处弯曲，磁压力使之继续弯

致稳

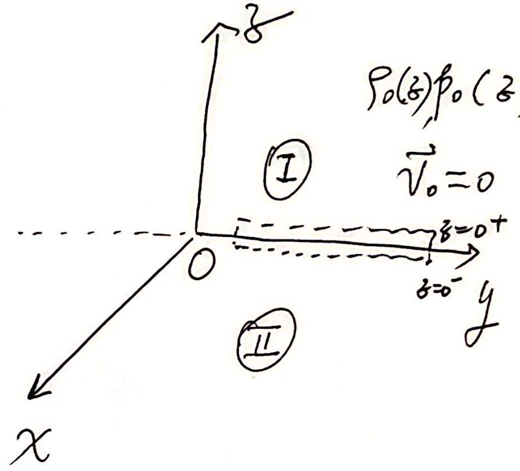


5.3 Rayleigh-Taylor 和 Kelvin-Helmholtz 不稳定性（头重脚轻 top-heavy 与速度剪切 Velocity shear）

查 <Bilibili 视频>.

5.3.1 RT 不稳定性：从流体力学到磁流体力学

双层流体、分别均匀，仅在分界面处有参数（密度）跃变，$\vec{g}=(0,0,-g)$



HD 方程组：

$$\begin{cases} \frac{\partial \rho}{\partial t}+\nabla\cdot(\rho\vec{v})=0 \\ \frac{\partial p}{\partial t}+\nabla\cdot(p\vec{v})+(\gamma-1)p\nabla\cdot\vec{v}=0 \\ \rho\frac{d\vec{v}}{dt}=-\nabla p+\rho\vec{g} \end{cases} \xRightarrow[\text{小量化}]{\text{小扰动线}} \begin{cases} \frac{\partial \rho'}{\partial t}+\rho_0\nabla\cdot\vec{v}+\vec{v}\cdot\nabla\rho_0=0 \\ \frac{\partial p'}{\partial t}+p_0\nabla\cdot\vec{v}+\vec{v}\cdot\nabla p_0+(\gamma-1)p_0\nabla\cdot\vec{v}=0 \\ \rho_0\frac{\partial \vec{v}}{\partial t}=-\nabla p'+\rho'\vec{g},\ \cancel{-\nabla p_0+\rho_0\vec{g}=0}\ (\text{不用考虑零阶压力平衡项}) \end{cases}$$

$$\begin{cases} p=p_0(z)+p' \\ \rho=\rho_0(z)+\rho' \end{cases}$$

Fourier Transform：注意 z 方向背景参量沿有变化，与以往不同，无法消掉 $\frac{\partial}{\partial z}\rho_0(z)$ 与 $\frac{\partial}{\partial z}p_0(z)$，可先保留

设扰动量 $\vec{v}(\vec{r},t)=v(z)\exp(ik_x x+ik_y y-i\omega t)$，$\vec{k}=k_x\hat{e}_x+k_y\hat{e}_y$，则 $\frac{\partial}{\partial t}\sim-i\omega$，

$$\begin{cases}\nabla\sim i\vec{k}+\frac{\partial}{\partial z}\hat{e}_z \\ \nabla\rho_0\sim\frac{\partial\rho_0}{\partial z}\hat{e}_z=\frac{d\rho_0}{dz}\hat{e}_z \\ \nabla p_0=\frac{dp_0}{dz}\hat{e}_z\end{cases}$$

考虑不可压情况（$\nabla\cdot\vec{v}=0$），可简化：，即有 $i\vec{k}\cdot\vec{v}+\frac{\partial v_z}{\partial z}=0$

$$\begin{cases} -i\omega\rho'+i\rho_0\vec{k}\cdot\vec{v}+\rho_0\frac{\partial v_z}{\partial z}+v_z\frac{d\rho_0}{dz}=0 \Rightarrow -i\omega\rho'+v_z\frac{d\rho_0}{dz}=0 \Rightarrow \rho'=-\frac{i}{\omega}v_z\frac{d\rho_0}{dz} \quad ① \\ -i\omega p'+v_z\frac{dp_0}{dz}=0 \Rightarrow p'=-\frac{i}{\omega}v_z\frac{dp_0}{dz} \quad ② \\ -i\omega\rho_0\vec{v}=-i\vec{k}p'-\frac{dp'}{dz}\hat{e}_z-\rho' g\hat{e}_z \Rightarrow \begin{cases}\omega\rho_0\vec{v}_k=\vec{k}p' \quad ③ \Rightarrow p'=i\frac{\omega\rho_0}{k^2}\frac{dv_z}{dz} \\ i\omega\rho_0 v_z=\frac{\partial p'}{\partial z}+\rho' g \quad ④\end{cases}\end{cases}$$

$\vec{v}=v_k\frac{\vec{k}}{k}+v_z\hat{e}_z$

由不可压条件，可得：

$kv_k=+i\frac{\partial v_z}{\partial z}$ ⓪

$=i\frac{dv_z}{dz}$.

联合以上方程①–④，可保留 v_z 项，得：

由 z 方向运动方程 ④ $\Rightarrow \omega^2\rho_0 v_z=\frac{\omega^2}{k^2}\left(\rho_0\frac{d^2v_z}{dz^2}+\frac{dv_z}{dz}\frac{d\rho_0}{dz}\right)-v_z g\frac{d\rho_0}{dz} \Rightarrow$ 在 I、II 区分别成立，$k^2v_z=d^2v_z/dz^2$

v_z 扰动幅度，仅依赖于 z，可设离 z=0 越远，【越小】越小，指数下降，$v_z\sim V_z\exp(\pm kz)$

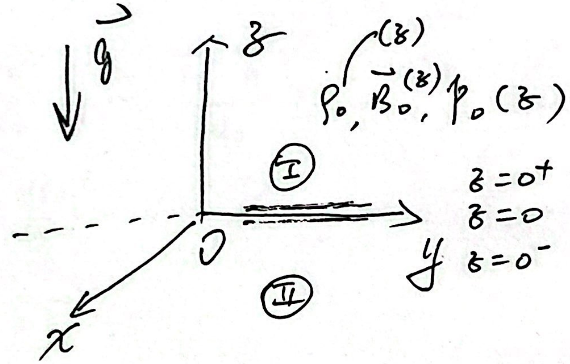
得 $v_z\sim V_z\exp(\pm kz)$，z>0 时，取"−"，z<0 时，取"+"，得 $\frac{dv_z}{dz}=\pm kv_z$，$\frac{d^2v_z}{dz^2}=k^2v_z$， ③

得 $\omega^2\rho_0 = \frac{\omega^2}{k^2}k^2\rho_0 \pm \frac{\omega^2}{k^2}k\frac{d\rho_0}{dz} - g\frac{d\rho_0}{dz}$，对 $z=0$ 上下薄层做线积分（由 $z=0^-$ 积至 $z=0^+$），得

$$0 = -\frac{\omega^2}{k^2}k\rho_{10} - \frac{\omega^2}{k^2}k\rho_{20} - g(\rho_{10}-\rho_{20}) \Rightarrow \frac{\omega^2}{k^2}k(\rho_{10}+\rho_{20}) = -g(\rho_{10}-\rho_{20}) \Rightarrow \omega^2 = -\frac{kg(\rho_{10}-\rho_{20})}{\rho_{10}+\rho_{20}} < 0$$

$\omega^2<0$，绝对不稳定.

$\boxed{\text{MHD RT：}}$



$\rho_0(z), \vec{B}_0(z), p_0(z), \vec{v}_0=0$

$\vec{g} = (0, 0, -g)$

$\vec{B}_0 = (B_x(z), B_y(z), 0)$

MHD方程组：

$$\begin{cases} \frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\vec{v}) = 0 \\ \frac{\partial p}{\partial t} + \nabla\cdot(p\vec{v}) + (\gamma-1)p\nabla\cdot\vec{v} = 0 \\ \rho\frac{d\vec{v}}{dt} = -\nabla p + \rho\vec{g} + \vec{j}\times\vec{B} = -\nabla(p+B^2/2\mu_0) + \frac{\vec{B}\cdot\nabla\vec{B}}{\mu_0} + \rho\vec{g} \\ \frac{\partial\vec{B}}{\partial t} = \nabla\times(\vec{v}\times\vec{B}) \qquad (\vec{E}+\vec{v}\times\vec{B}=0). \\ \nabla\cdot\vec{B}=0 \end{cases}$$

z方向是一个特殊的方向，在该方向上背景参数不均匀，$\frac{\partial\rho_0}{\partial z} = \frac{d\rho_0(z)}{dz} \neq 0$，FT时需保留 $\frac{d}{dz}\hat{e}_z$.

对 x、y 方向仍可用简谐振荡表示，$e^{i\vec{k}\cdot\vec{r}-i\omega t}$

$\vec{k} = k_x\hat{e}_x + k_y\hat{e}_y$.

$\nabla \sim i\vec{k} + \frac{d}{dz}\hat{e}_z$，$\frac{\partial}{\partial t} \sim -i\omega$.
（作用于扰动量）.

$\nabla\rho_0 \sim \frac{d}{dz}\rho_0\hat{e}_z$.（作用于背景参量）

小扰动、线性化，得：

$$\begin{cases} \frac{\partial\rho'}{\partial t} + \rho_0\nabla\cdot\vec{v} + \vec{v}\cdot\nabla\rho_0 = 0 \\ \frac{\partial p'}{\partial t} + \gamma p_0\nabla\cdot\vec{v} + \vec{v}\cdot\nabla p_0 = 0 \\ \frac{\partial\vec{B}'}{\partial t} = -\vec{B}_0\nabla\cdot\vec{v} + \vec{B}_0\cdot\nabla\vec{v} - \vec{v}\cdot\nabla\vec{B}_0 \\ \nabla\cdot\vec{B}'=0 \\ \rho_0\frac{\partial\vec{v}}{\partial t} = -\nabla(p'+\frac{\vec{B}_0\cdot\vec{B}'}{\mu_0}) + \frac{\vec{B}_0\cdot\nabla\vec{B}'+\vec{B}'\cdot\nabla\vec{B}_0}{\mu_0} + \rho'\vec{g} \end{cases}$$

，$-\nabla p_0 + \rho_0\vec{g} - \nabla\frac{B^2}{2\mu_0} + \frac{\vec{B}_0\cdot\nabla\overset{=0}{\vec{B}_0}}{\mu_0} = 0$

④

假设不可压，即 $\nabla\cdot\vec{v}=0$，得 $\vec{k}\cdot\vec{v}=+i\frac{dv_z}{dz}$，$v_k k = i\frac{dv_z}{dz}$ ……① ；又由 $\nabla\cdot\vec{B}'=0$，得 $\vec{k}\cdot\vec{B}'=i\frac{dB_z}{dz}$ ……②

有：$i\omega\rho' = v_z\frac{d\rho_0}{dz}$ 或 $\rho' = -\frac{iv_z}{\omega}d\rho_0/dz$ ---③ ，$i\omega p' = v_z\frac{dp_0}{dz}$，或 $p' = -\frac{iv_z}{\omega}dp_0/dz$ ……④

$i\omega\vec{B}' = v_z\frac{d\vec{B}_0}{dz} - i\vec{k}\cdot\vec{B}_0\vec{v} \Rightarrow \vec{B}' = -\frac{iv_z}{\omega}\frac{d\vec{B}_0}{dz} - \frac{\vec{k}\cdot\vec{B}_0}{\omega}\vec{v}$ ……⑤ $\Rightarrow B_z' = -\frac{\vec{k}\cdot\vec{B}_0}{\omega}v_z$ ……⑥

$$\omega\rho_0\vec{v} = \vec{k}\left(p'+\frac{\vec{B}_0\cdot\vec{B}'}{\mu_0}\right) - \frac{\vec{B}_0\cdot\vec{k}}{\mu_0}\vec{B}' + \frac{iB_z'\frac{d\vec{B}_0}{dz}}{\mu_0} - i\frac{d}{dz}\left(p'+\frac{\vec{B}_0\cdot\vec{B}'}{\mu_0}\right)\hat{e}_z - i\rho' g\hat{e}_z$$

其中 $\vec{k}\left(p'+\frac{\vec{B}_0\cdot\vec{B}'}{\mu_0}\right) = -\frac{iv_z}{\omega}\frac{dp_0}{dz}\vec{k} - \frac{iv_z}{\omega\mu_0}\left(\frac{d\vec{B}_0}{dz}\cdot\vec{B}_0\right)\vec{k} - \frac{(\vec{k}\cdot\vec{B}_0)}{\omega\mu_0}(\vec{v}\cdot\vec{B}_0)\vec{k}$，$-\frac{\vec{B}_0\cdot\vec{k}}{\mu_0}\vec{B}' = \frac{iv_z}{\mu_0\omega}\frac{d\vec{B}_0}{dz}(\vec{B}_0\cdot\vec{k}) + \frac{(\vec{k}\cdot\vec{B}_0)^2}{\mu_0\omega}\vec{v}$

注意到 $\frac{iB_z'\frac{d\vec{B}_0}{dz}}{\mu_0} = -i\frac{\vec{k}\cdot\vec{B}_0}{\omega\mu_0}\frac{d\vec{B}_0}{dz}v_z$ 与 $-\frac{\vec{B}_0\cdot\vec{k}}{\mu_0}\vec{B}'$ 中的第一项抵消，将之第二项移至左侧，得：

$$\left(\omega^2-\frac{(\vec{k}\cdot\vec{B}_0)^2}{\mu_0\rho_0}\right)\vec{v} = -\frac{iv_z}{\rho_0}\frac{dp_0}{dz}\vec{k} - \frac{iv_z}{\mu_0\rho_0}\left(\frac{d\vec{B}_0}{dz}\cdot\vec{B}_0\right)\vec{k} - \frac{(\vec{k}\cdot\vec{B}_0)(\vec{v}\cdot\vec{B}_0)}{\mu_0\rho_0}\vec{k} - \hat{e}_z\frac{d}{dz}\left[\frac{v_z}{\rho_0}\frac{dp_0}{dz} + \frac{v_z}{\mu_0\rho_0}\frac{d\vec{B}_0}{dz}\cdot\vec{B}_0 - \frac{i(\vec{k}\cdot\vec{B}_0)(\vec{v}\cdot\vec{B}_0)}{\mu_0\rho_0}\right] - \frac{v_z}{\rho_0}\frac{d\rho_0}{dz}g\hat{e}_z$$

可将之拆分成 v_k 分量与 v_z 分量，注意 $v_k = -\frac{i}{k}\frac{dv_z}{dz}$，得：

$$\begin{cases}\left(\omega^2-(\vec{k}\cdot\vec{v}_A)^2\right)i\frac{dv_z}{dz} = \left[\frac{iv_z}{\rho_0}\frac{dp_0}{dz} + i\frac{v_z}{\mu_0\rho_0}\left(\frac{d\vec{B}_0}{dz}\cdot\vec{B}_0\right) + \frac{(\vec{k}\cdot\vec{B}_0)(\vec{v}\cdot\vec{B}_0)}{\mu_0\rho_0}\right]k^2 \\ \left(\omega^2-(\vec{k}\cdot\vec{v}_A)^2\right)v_z = -\frac{d}{dz}\left[\frac{v_z}{\rho_0}\frac{dp_0}{dz} + \frac{v_z}{\mu_0\rho_0}\frac{d\vec{B}_0}{dz}\cdot\vec{B}_0 - \frac{i(\vec{k}\cdot\vec{B}_0)(\vec{v}\cdot\vec{B}_0)}{\mu_0\rho_0}\right] - \frac{v_z}{\rho_0}\frac{d\rho_0}{dz}g\end{cases}$$

将上式代入下式，得 $\left[\omega^2-(\vec{k}\cdot\vec{v}_A)^2\right]v_z = +\frac{d}{dz}\left\{\left[\omega^2-(\vec{k}\cdot\vec{v}_A)^2\right]\frac{dv_z}{dz}\right\}/k^2 - \frac{v_z}{\rho_0}\frac{d\rho_0}{dz}g$

上式在Ⅰ、Ⅱ均成立，则在Ⅰ、Ⅱ区有：$\left[\omega^2-(\vec{k}\cdot\vec{v}_A)^2\right]\left(v_z k^2-\frac{d^2v_z}{dz^2}\right)=0$，即在Ⅰ、Ⅱ区均有 $\frac{d^2v_z}{dz^2}=k^2v_z$.

同样，离 $z=0$ 越远，振幅越小，则

$v_z = v_z\exp(\mp kz)$，$z=0^+$ 时，取"-"，$z=0^-$ 时，取"+"，则对 $z=0$ 面上下薄层积分，得：

⑤

在边界层处，满足 $\left[\cancel{\omega^2=(\vec k\cdot\vec V_A)}\right]$ $\left[\omega^2-(\vec k\cdot\vec V_A)^2\right]k^2 v_z = \frac{d}{dz}\left[\omega^2-(\vec k\cdot\vec V_A)^2\right]\frac{dv_z}{dz} + \left[\omega^2-(\vec k\cdot\vec V_A)^2\right]\frac{d^2 v_z}{dz^2} - \frac{k^2 g}{\rho_0} v_z \frac{d\rho_0}{dz}$

将 v_z 表达式代入，得：$\frac{d}{dz}\left[\rho_0\omega^2 - \frac{(\vec k\cdot\vec B_0)^2}{\mu_0}\right](\pm k) - k^2 g\frac{d\rho_0}{dz} = 0$

对 z 积分，得 $\left[\rho_{10}\omega^2 - \frac{(\vec k\cdot\vec B_{10})^2}{\mu_0}\right](-k) - k^2 g\rho_{10} = \left[\rho_{20}\omega^2 - \frac{(\vec k\cdot\vec B_{20})^2}{\mu_0}\right]k - k^2 g\rho_{20}$

得：$\omega^2 = -\frac{kg(\rho_{10}-\rho_{20})}{\rho_{20}+\rho_{10}} + \frac{(\vec k\cdot\vec B_{10})^2+(\vec k\cdot\vec B_{20})^2}{\mu_0(\rho_{20}+\rho_{10})}$ （磁力 VS 重力）

剪切

可知，磁场起致稳作用，若 $\vec k\perp\vec B_0$，则无作用；有条件稳定. $\rho_{10}=\rho_{20}$ 时. $\omega^2=(\vec k\cdot\vec V_A)^2$，Alfvén 波.

5.3.2 KH 不稳定性：从流体力学到磁流体力学

不考虑重力. $\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\vec v)=0$

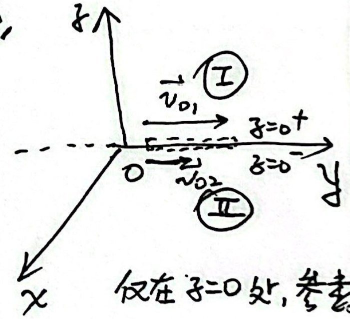
$\nabla\cdot\vec v=0$
不可压

$$\begin{cases}\frac{\partial p}{\partial t}+\nabla\cdot(p\vec v)+(\gamma-1)p\nabla\cdot\vec v=0\\ \rho\frac{d\vec v}{dt}=-\nabla p=\rho\left(\frac{\partial\vec v}{\partial t}+\vec v\cdot\nabla\vec v\right).\end{cases}$$

$\vec v=\vec v_0+\vec v'$

$\rho=\rho_0+\rho'$

$p=p_0+p'$



仅在 $z=0$ 处，参数有跳变

交换不稳定性 Interchange Instab.

B ρ $\vec F=-\mu\nabla B$ $\vec F$

不稳定.

B ρ

（等效重力）.

$$\begin{cases}\frac{\partial\rho'}{\partial t}+\vec v'\cdot\nabla\rho_0+\vec v_0\cdot\nabla\rho'+\underbrace{\rho'\nabla\cdot\vec v_0+\rho_0\nabla\cdot\vec v'}_{=0}=0\\ \rho_0\left(\frac{\partial\vec v'}{\partial t}+\vec v_0\cdot\nabla\vec v'+\vec v'\cdot\nabla\vec v_0\right)=-\nabla p',\\ \frac{\partial p'}{\partial t}+\underbrace{\gamma p'\nabla\cdot\vec v_0+\gamma p_0\nabla\cdot\vec v'}_{=0}+\vec v_0\cdot\nabla p'+\vec v'\cdot\nabla p_0=0\end{cases}$$

$\rho_0\vec v_0\cdot\nabla\vec v_0+\nabla p_0=0\Rightarrow\nabla p_0=0$，$p_{0\text{I}}=p_{0\text{II}}$.

$\nabla p_0=\frac{dp_0}{dz}\hat e_z$. 上下界面压力平衡条件.

将扰动量写成 $(\rho',p',\vec v')\sim(\rho',p',\vec v')\exp\left[i(\vec k\cdot\vec r-\omega t)\pm k_z z\right]$，$z=0^+$ 时，取 −；$z=0^-$ 时，取 +

结合适当的边界面处物理条件，可以只需处理 z 方向运动方程.　⑥

$$\rho_0\left(\frac{\partial v_z'}{\partial t}+\vec{v}_0\cdot\nabla v_z'\right)=-\frac{\partial p'}{\partial z} \Rightarrow -i\rho_0(\omega-\vec{k}\cdot\vec{v}_0)v_z'=-\frac{\partial p'}{\partial z}$$

① 分界面处位置移量表示：$\xi=\xi_0\exp(i(\vec{k}\cdot\vec{r}-\omega t))$，得 $v_z'=\frac{d\xi}{dt}=\left(\frac{\partial}{\partial t}+\vec{v}_0\cdot\nabla\right)\xi=-i(\omega-\vec{k}\cdot\vec{v}_0)\xi$

将速度用

得 $-\rho_0(\omega-\vec{k}\cdot\vec{v}_0)^2\xi=-\frac{\partial p'}{\partial z}$

② 压强连续，在分界面处取无穷薄片，体积分，可知体积内质量 → 0，故与密度相关项 → 0，仅余下压强.

分界面处

得：$-\rho_{10}(\omega-\vec{k}\cdot\vec{v}_{10})^2\xi_1=\rho_{20}(\omega-\vec{k}\cdot\vec{v}_{20})^2\xi_2$，$\boxed{\text{界面位移量连续}}$，$\xi_1=\xi_2$，得

$$\rho_{10}\left[\omega^2-2\omega\vec{k}\cdot\vec{v}_{10}+(\vec{k}\cdot\vec{v}_{10})^2\right]+\rho_{20}\left[\omega^2-2\omega\vec{k}\cdot\vec{v}_{20}+(\vec{k}\cdot\vec{v}_{20})^2\right]=0$$

$$\omega^2(\rho_{10}+\rho_{20})-2\omega\left[\rho_{10}\vec{k}\cdot\vec{v}_{10}+\rho_{20}\vec{k}\cdot\vec{v}_{20}\right]+\rho_{10}(\vec{k}\cdot\vec{v}_{10})^2+\rho_{20}(\vec{k}\cdot\vec{v}_{20})^2=0$$

得：$$\omega=\frac{\rho_{10}\vec{k}\cdot\vec{v}_{10}+\rho_{20}\vec{k}\cdot\vec{v}_{20}\pm i\sqrt{\rho_{10}\rho_{20}(\vec{k}\cdot\vec{v}_{10}-\vec{k}\cdot\vec{v}_{20})^2}}{\rho_{10}+\rho_{20}}$$

Ⅰ p', p_0　$z=0^+$

Ⅱ p'　$z=0$

$\frac{\partial p_{Ⅰ}'}{\partial z}\sim\frac{p'-p_0}{\Delta z}$

$\frac{\partial p_{Ⅱ}'}{\partial z}\sim\frac{p_0-p'}{\Delta z}$

二者相反，大小相等.

同样是绝对不稳定，$\vec{v}_{10}=\vec{v}_{20}$，$\rho_{10}=\rho_{20}$ 时，$\omega=\vec{k}\cdot\vec{v}_0$，Doppler 频移.

$\boxed{\text{MHD}-\text{KH}}$：
(不可压)

$$\rho_0\left(\frac{\partial\vec{v}'}{\partial t}+\vec{v}_0\cdot\nabla\vec{v}'+\vec{v}'\cdot\nabla\vec{v}_0\right)=-\nabla\left(p'+\frac{\vec{B}_0\cdot\vec{B}'}{\mu_0}\right)+\frac{\vec{B}_0\cdot\nabla\vec{B}'}{\mu_0}+\frac{\vec{B}'\cdot\nabla\vec{B}_0}{\mu_0}$$

z 方向分量：$$\rho_0\left(\frac{\partial v_z'}{\partial t}+i\vec{k}\cdot\vec{v}_0v_z'\right)=-\frac{\partial}{\partial z}\left(p'+\frac{\vec{B}_0\cdot\vec{B}'}{\mu_0}\right)+\frac{i\vec{k}\cdot\vec{B}_0B_z'}{\mu_0}$$

$\boxed{\vec{k}\text{与}\vec{B}_0\text{均位于}xoy\text{面}}$

$$-\rho_0(\omega-\vec{k}\cdot\vec{v}_0)^2\xi-\frac{i\vec{k}\cdot\vec{B}_0B_z'}{\mu_0}=-\frac{\partial}{\partial z}\left(p'+\frac{\vec{B}_0\cdot\vec{B}'}{\mu_0}\right)$$

由 $\frac{\partial\vec{B}}{\partial t}=\nabla\times(\vec{v}\times\vec{B})=\vec{B}\cdot\nabla\vec{v}-\vec{v}\cdot\nabla\vec{B}$，得 $\frac{\partial\vec{B}'}{\partial t}=\vec{B}_0\cdot\nabla\vec{v}'+\vec{B}'\cdot\nabla\vec{v}_0-\vec{v}_0\cdot\nabla\vec{B}'-\vec{v}'\cdot\nabla\vec{B}_0$，$-i\omega B_z'=i\vec{B}_0\cdot\vec{k}v_z'-i\vec{v}_0\cdot\vec{k}B_z'\Rightarrow(\omega-\vec{k}\cdot\vec{v}_0)B_z'=-\vec{k}\cdot\vec{B}_0$

得: $\left.\begin{aligned} B_z' &= \dfrac{-\vec{k}\cdot\vec{B_0}\, v_z'}{\omega-\vec{k}\cdot\vec{v}_0} \\ B_z' &= i\vec{k}\cdot\vec{B_0}\,\xi \end{aligned}\right\} \Rightarrow -\rho_0(\omega-\vec{k}\cdot\vec{v}_0)^2\xi + \dfrac{(\vec{k}\cdot\vec{B_0})^2}{\mu_0}\xi = -\dfrac{\partial}{\partial z}\left(p' + \dfrac{\vec{B_0}\cdot\vec{B}'}{\mu_0}\right)$ ⑦

同样基于压强连续条件,得: $-\rho_{10}(\omega-\vec{k}\cdot\vec{v}_{10})^2 + \dfrac{(\vec{k}\cdot\vec{B}_{10})^2}{\mu_0} = \rho_{20}(\omega-\vec{k}\cdot\vec{v}_{20})^2 - \left(\dfrac{\vec{k}\cdot\vec{B}_{20}}{\mu_0}\right)^2$.

得 $\rho_{10}(\omega-\vec{k}\cdot\vec{v}_{20})^2 + \rho_{20}(\omega-\vec{k}\cdot\vec{v}_{20})^2 - \dfrac{(\vec{k}\cdot\vec{B}_{10})^2}{\mu_0} - \dfrac{(\vec{k}\cdot\vec{B}_{20})^2}{\mu_0} = 0$.

$$\Rightarrow \omega = \frac{\rho_{10}\vec{k}\cdot\vec{v}_{10} + \rho_{20}\vec{k}\cdot\vec{v}_{20} \pm \sqrt{\frac{\rho_{10}+\rho_{20}}{\mu_0}\left[(\vec{k}\cdot\vec{B}_{10})^2+(\vec{k}\cdot\vec{B}_{20})^2\right] - \rho_{10}\rho_{20}(\vec{k}\cdot\vec{v}_{10}-\vec{k}\cdot\vec{v}_{20})^2}}{\rho_{10}+\rho_{20}}.$$

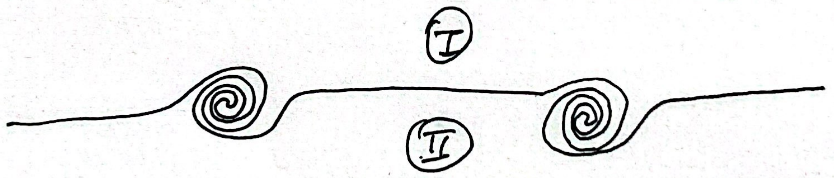
可知: ① $\vec{v}_{10}=\vec{v}_{20}$, $\vec{B}_{10}=\vec{B}_{20}$ 时, $\omega = \vec{k}\cdot(\vec{v}_0+\vec{v}_A)$, Alfvén 波的增长.

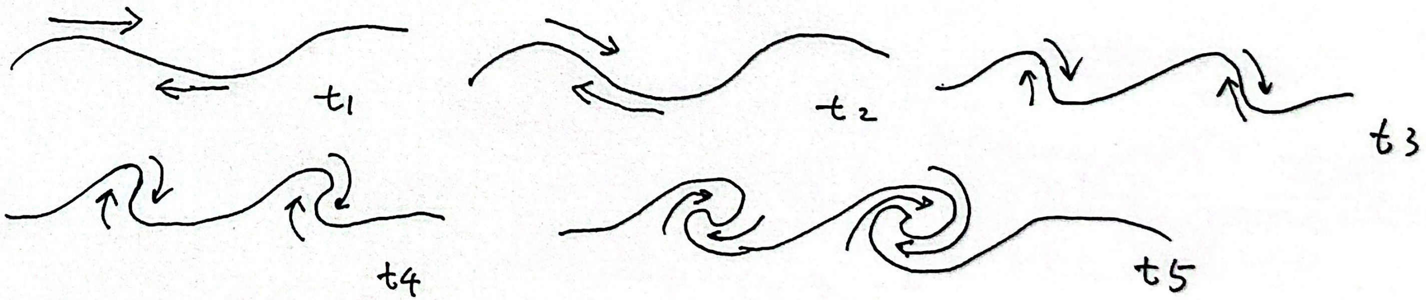
② 磁场起致稳作用,有条件稳定,可易得稳定条件: $\dfrac{\rho_{10}+\rho_{20}}{\mu_0}\left[(\vec{k}\cdot\vec{B}_{10})^2+(\vec{k}\cdot\vec{B}_{20})^2\right] \geq \rho_{10}\rho_{20}(\vec{k}\cdot\vec{v}_{10}-\vec{k}\cdot\vec{v}_{20})^2$

③ $\vec{k}\perp\vec{B}_0$ 时,磁场无影响.

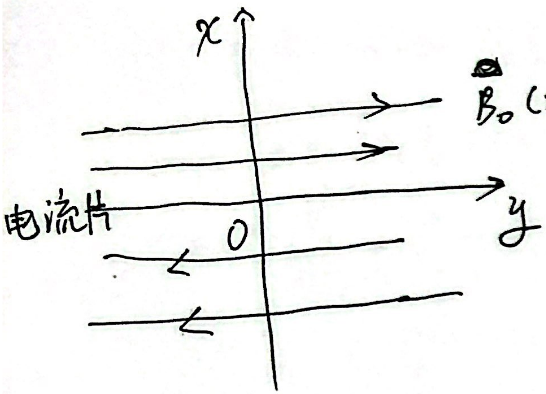
④ $\left.\begin{aligned}\vec{k}\parallel\vec{B}_0 \\ \vec{k}\parallel\vec{v}_0\end{aligned}\right\}$, $\rho_{10}=\rho_{20}$ 时. 不稳定条件为: $\Delta v \geq 2v_A$.

⑤ 典型涡旋结构:





5.4 撕裂模不稳定性 (Tearing-mode Instability) —— 磁岛的形成 (Magnetic Island)



$\vec{B}_0(x)\,\hat{e}_y$，背景磁场在 $x=0$ 处，反向，形成电流片．设不可压，不考虑热效应，有

$$\rho\frac{d\vec{v}}{dt}=\vec{j}\times\vec{B},\quad \frac{\partial\vec{B}}{\partial t}=-\nabla\times\vec{E},\quad \vec{E}+\vec{v}\times\vec{B}=\eta\vec{j},\quad \nabla\cdot\vec{v}=0,\quad \nabla\cdot\vec{B}=0.$$

设扰动参量 $A(x,y,t)=A(x)e^{\gamma t}e^{iky}$，则 $\frac{\partial}{\partial t}\sim\gamma$，$\nabla\sim ik\hat{e}_y+\frac{\partial}{\partial x}\hat{e}_x$

$\frac{\partial}{\partial z}=0$ (二维)

$\vec{k}=k\hat{e}_y$．

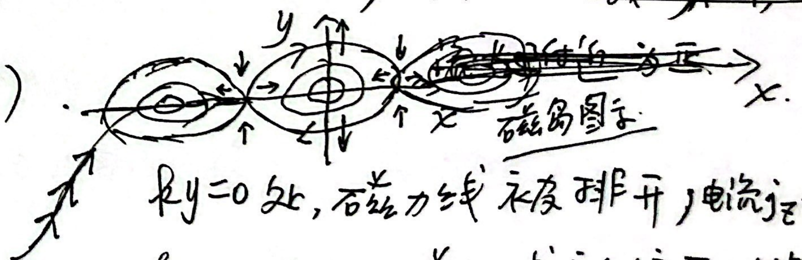
得：$\gamma\rho v_x=-j_zB_0$，$\gamma B_x=-ikE_z$，$E_z+v_xB_0=\eta j_z$，$ikv_y=-\frac{\partial v_x}{\partial x}$，$ikB_y=-\frac{\partial B_x}{\partial x}$．

共六个扰动量 $(v_x, v_y, B_x, B_y, E_z, j_z)$，均可用 B_x 表示：

$$B_y=\frac{i}{k}\frac{\partial B_x}{\partial x},\quad E_z=\frac{i\gamma B_x}{k},\quad v_x=-\frac{i\gamma B_xB_0}{k(\eta\gamma\rho+B_0^2)},\quad j_z=-\frac{\gamma\rho v_x}{B_0}=\frac{i\rho\gamma^2B_x}{k(\eta\gamma\rho+B_0^2)},\quad v_y=\frac{\gamma}{k^2}\frac{\partial}{\partial x}\left[\frac{B_xB_0}{\eta\gamma\rho+B_0^2}\right]$$

主要观察 v_x．v_y 与 B_x 认识磁岛形成，注意：$x>0$ 时，$B_0>0$；$x<0$ 时．$B_0<0$，

可设 $B_x=|B_x|e^{\gamma t}e^{iky+i\frac{\pi}{2}}=|B_x|e^{\gamma t}(-\sin ky+i\cos ky)$．



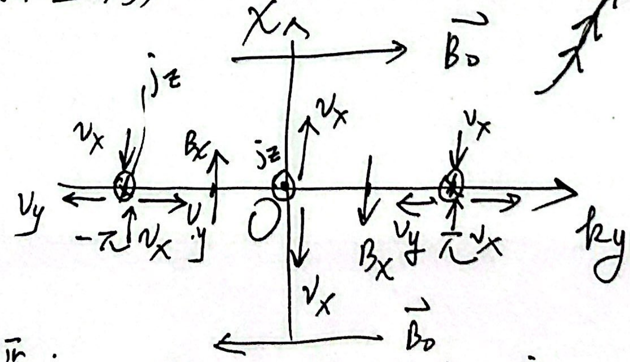
得各扰动量实部 (观察 $ky=\pm\pi,\pm\frac{\pi}{2},0$ 时取值符号)

$B_x\sim-\sin ky$

$v_x\sim B_0\cos ky$

$v_y\sim-\sin ky\,\frac{\partial}{\partial x}\underbrace{\left[\frac{|B_x|B_0}{\eta\gamma\rho+B_0^2}\right]}_{\substack{x>0\text{时，为正.}\\x<0\text{时，为负}}}$ } 梯度为正．

$j_z\sim-\cos ky$



B_x，v_x 与 v_y 方向示意

$ky=0$ 处，磁力线被排开，电流 j_z 向外；

$ky=\pm\pi$ 处，磁力线被挤压，电流 j_z 方向与 $\vec{B}_0$ 电流方向相同，电流增强，

(文拉)

有限电阻情况下，可触发重联，切割力线，力线被割开后，向中心收缩，呈岛状．

5.5 束流不稳定性（streaming instability）.

不同成份之间有流速差时，可激发波不稳定性。考虑静电、非磁化情况，多元流体方程组：

$$\left.\begin{aligned}
&\frac{\partial n_\alpha}{\partial t}+\nabla\cdot(n_\alpha\vec{v}_\alpha)=0\\
&n_\alpha m_\alpha\left(\frac{\partial\vec{v}_\alpha}{\partial t}+\vec{v}_\alpha\cdot\nabla\vec{v}_\alpha\right)=-\nabla p_\alpha+n_\alpha q_\alpha\vec{E}\\
&p_\alpha=C_\alpha\rho_\alpha^{\gamma_\alpha}\\
&\nabla\cdot\vec{E}=\sum_\alpha n_\alpha q_\alpha/\varepsilon_0
\end{aligned}\right\}$$

① 小扰动 $n_\alpha=n_{\alpha0}+n_\alpha'$，$\sum_\alpha n_{\alpha0}q_\alpha=0$

$\vec{v}_\alpha=\vec{v}_{\alpha0}+\vec{v}_\alpha'$，$p_\alpha=p_{\alpha0}+p_\alpha'$

② 线性化

$$\left.\begin{aligned}
&\frac{\partial n_\alpha'}{\partial t}+\vec{v}_{\alpha0}\cdot\nabla n_\alpha'+n_{\alpha0}\nabla\cdot\vec{v}_\alpha'=0\\
&n_{\alpha0}m_\alpha\left(\frac{\partial\vec{v}_\alpha'}{\partial t}+\vec{v}_{\alpha0}\cdot\nabla\vec{v}_\alpha'\right)=-\nabla p_\alpha'+n_{\alpha0}q_\alpha\vec{E}\\
&p_\alpha'=\rho_\alpha'c_{s\alpha}^2,\quad c_{s\alpha}^2=\frac{\gamma_\alpha p_{\alpha0}}{\rho_{\alpha0}}\\
&\nabla\cdot\vec{E}=\sum_\alpha n_\alpha'q_\alpha/\varepsilon_0
\end{aligned}\right\}$$

③ FT

$$\Rightarrow\begin{cases}
(\omega-\vec{k}\cdot\vec{v}_{\alpha0})n_\alpha'=n_{\alpha0}\vec{k}\cdot\vec{v}_\alpha'\\
(\omega-\vec{k}\cdot\vec{v}_{\alpha0})\vec{v}_\alpha'=\dfrac{\vec{k}c_{s\alpha}^2n_\alpha'}{n_{\alpha0}}+\dfrac{iq_\alpha\vec{E}}{m_\alpha}\\
i\vec{k}\cdot\vec{E}=\sum_\alpha n_\alpha'q_\alpha/\varepsilon_0=\sum_\alpha\dfrac{n_{\alpha0}\vec{k}\cdot\vec{v}_\alpha'}{\omega-\vec{k}\cdot\vec{v}_{\alpha0}}\dfrac{q_\alpha}{\varepsilon_0}
\end{cases}$$

$$\Rightarrow\left\{\frac{\vec{k}\cdot\vec{v}_\alpha'}{\omega-\vec{k}\cdot\vec{v}_{\alpha0}}\left[(\omega-\vec{k}\cdot\vec{v}_{\alpha0})^2-k^2c_{s\alpha}^2\right]=\frac{iq_\alpha\vec{k}\cdot\vec{E}}{m_\alpha}\right.$$

$$\Rightarrow 1=\sum_\alpha\frac{\omega_{p\alpha}^2}{(\omega-\vec{k}\cdot\vec{v}_{\alpha0})^2-k^2c_{s\alpha}^2}$$

考虑电子、质子等离子体（$\vec{v}_{e,p0}=0$）及束流电子（$T_{b0}=0$，$\vec{v}_{b0}$，n_{b0}，$\alpha=n_{b0}/n_{e0}$），得：

$$1=\frac{\omega_{pe}^2}{\omega^2-k^2c_{se}^2}+\frac{\omega_{pi}^2}{\omega^2-k^2c_{si}^2}+\frac{\alpha\omega_{pe}^2}{(\omega-\vec{k}\cdot\vec{v}_{b0})^2}.$$

对于静电波，可设 $\vec{k}\parallel\vec{v}_{b0}$.

① $\alpha=0$ 时，Langmuir 波色散关系.

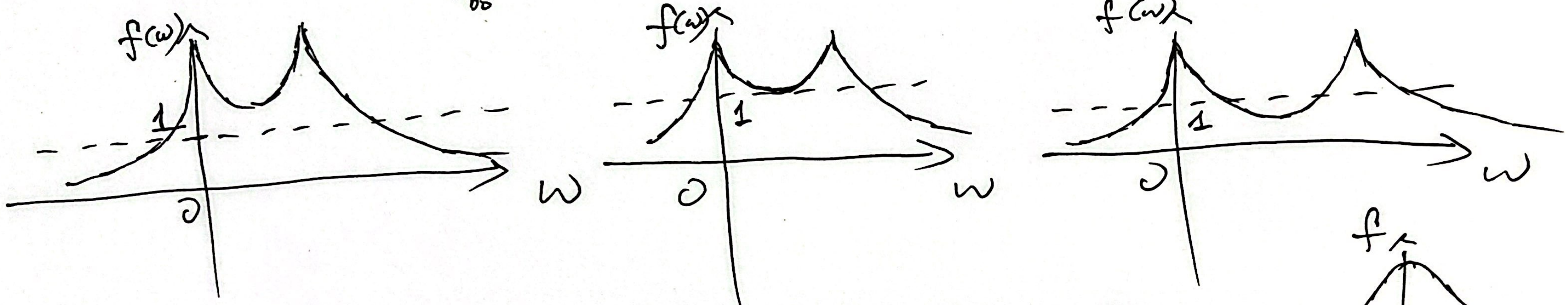
② 可忽略质子贡献，得 $1=\dfrac{\omega_{pe}^2}{\omega^2-k^2c_{se}^2}+\dfrac{\alpha\omega_{pe}^2}{(\omega-kv_{b0})^2}$

可设 $\omega\gg kc_{se}$，得：令 $f(\omega)=$ 上式右侧.

$$\begin{cases}\omega\to0,\ 及\ \omega\to kv_{b0}\ 时.\ f(\omega)\to\infty\\ \omega\to\pm\infty\ 时.\ f(\omega)\to0.\end{cases}$$

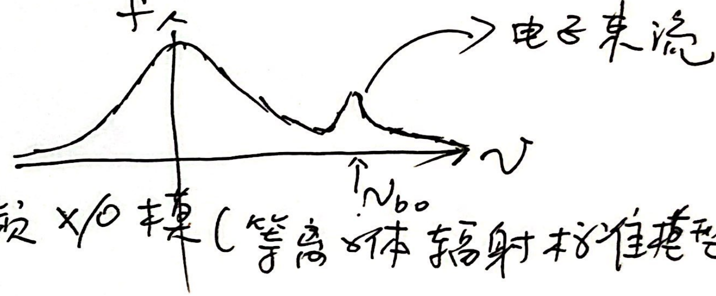
由 $\frac{df}{d\omega}=0$，可求得 $\omega_m=\frac{kv_{b0}}{1+\alpha^{\frac{1}{3}}}$，得 $f(\omega_m)=\frac{\omega_{pe}^2}{k^2v_{b0}^2}(1+\alpha^{\frac{1}{3}})^3$，当 $f(\omega_m)>1$ 时，存在虚根，即不稳定性。

得 $k^2=\left(\frac{2\pi}{\lambda}\right)^2<\frac{\omega_{pe}^2(1+\alpha^{\frac{1}{3}})^3}{v_{b0}^2}$，即 k 较小（λ 较大）时，更利于不稳定性的发生。 (10)



三种情况

动理论中，对应于尾瘤不稳定性，相应速度分布函数为：



可激发静电Langmuir波，波波作用可进一步导致基频和谐频X/O模（等离子体辐射标准模型）。